Chapter one : Linear Equations and Matrices

1.1 Systems of Linear Equations.

One of the most frequently recurring practical problems in many fields of study such as Mathematics, Physics, Biology, Chemistry, Economics, all phases of engineering, Operation research, and the social sciences is that of solving a system of linear equations. The equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b, \qquad \cdots (1)$$

which expresses the real or complex quantity b in terms of the unknowns $x_1, x_2, \cdots, x_n$ and the real or complex constants $a_1, a_2, \cdots, a_n$, is called a linear equation.

In many applications we are given b and must find numbers $x_1, x_2, \cdots, x_n$ satisfying (1).

A solution to linear equation (1) is a sequence of n numbers $s_1, s_2, \cdots, s_n$ which has the property that (1) is satisfied when $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$ are substituted in (1).

Example (1): $x_1 = 2$, $x_2 = 3$, and $x_3 = -4$ is a solution to the linear equation

$$6x_1 - 3x_2 + 4x_3 = -13,$$

because

$$6(2) - 3(3) + 4(-4) = -13.$$

More generally, a system of m linear equations in n unknowns, $x_1, x_2, \ldots, x_n$, or a linear system, is a set of m linear equations each in n unknowns. A linear system can conveniently be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\quad \vdots \qquad &\cdots (2) \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Thus the ith equation is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i.$$

In (2) the a_{ij} are known constants. Given values of $b_1, b_2, \ldots, b_m$, we want to find values of $x_1, x_2, \ldots, x_n$ that will satisfy each equation in (2).

A solution to linear system (2) is a sequence of n numbers $s_1, s_2, \dots, s_n$, which has the property that each equation in (2) is satisfied when $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ are substituted.

Remark 1: If the linear system (2) has no solution, it is said to be inconsistent; if it has a solution, it is called consistent.

Remark 2: If $b_1 = b_2 = \dots = b_m = 0$, then (2) is called a homogenous system.

Remark 3: Note that $x_1 = x_2 = \dots = x_n = 0$, then (2) is always a solution to a homogenous system, it is called the trivial solution.

Remark 4: A solution to a homogenous system in which not all of $x_1, x_2, \dots, x_n$ are zero is called a nontrivial solution.

* Consider another system of r linear equations in n unknowns:

$$\begin{aligned} c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n &= d_1 \\ c_{21}x_1 + c_{22}x_2 + \dots + c_{2n}x_n &= d_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots & \\ c_{r1}x_1 + c_{r2}x_2 + \dots + c_{rn}x_n &= d_r \end{aligned} \qquad \dots (3)$$

We say that (2) and (3) are equivalent if they both have exactly the same solutions.

Example 2: The linear system

$$\begin{aligned} x_1 - 3x_2 &= -7 \\ 2x_1 + x_2 &= 7 \end{aligned} \quad \cdots (4)$$

has only the solution $x_1 = 2$ and $x_2 = 3$.

Example 3: The linear system

$$\begin{aligned} 8x_1 - 3x_2 &= 7 \\ 3x_1 - 2x_2 &= 0 \\ 10x_1 - 2x_2 &= 14 \end{aligned} \quad \cdots (5)$$

also has only the solution $x_1 = 2$ and $x_2 = 3$.

Remark 5: The linear systems (4) and (5) are equivalent.

* To find a solution to a linear system, we shall use a technique called the method of elimination (elimination method); that is, we eliminate some variables by adding a multiple of one equation to another equation. Elimination merely amounts to the development of a new linear system that is equivalent to the original system, but is much simpler to solve.

* In this course we shall broaden our outlook by dealing with systems in which we have $m = n$, $m < n$, and $m > n$.

In this section we use the method of elimination as it was studied in high school. We shall look at this method in a much more systematic manner.

Example 4: Consider the linear system

$$\begin{aligned} x - 3y &= -7 \\ 2x - 6y &= 7. \end{aligned} \qquad \text{--- (6)}$$

Find the solution of the linear system (6).

Solution: We decide to eliminate x. We add (-2) times the first equation to the second one, i.e.

$$\begin{aligned} -2(x - 3y &= -7) \\ 2x - 6y &= 7 \end{aligned} \qquad \text{...(7)}$$

$$\begin{aligned} -2x + 6y &= 14 \\ 2x - 6y &= 7. \end{aligned} \qquad \text{... (8)}$$

We obtain $0 = 21$, which makes no sence. This means that (6) has no solution; it is inconsistent.

Example 5: Consider the linear system

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2. \end{aligned} \quad \text{--- (*)}$$

To eliminate x, we obtain

$$-2\,(x + 2y + 3z = 16)$$
$$2x - 3y + 2z = 14 \quad \text{--- (1)}$$

$$-3\,(x + 2y + 3z = 6)$$
$$3x + y - z = -2 \quad \text{--- (2)}$$

From Eq. (1) and (2), we have

$$\begin{aligned} -7y - 4z &= 2 \\ -5y - 10z &= -20 \end{aligned} \quad \text{--- (3)}$$

$$-7y - 4z = 2$$
$$-\tfrac{1}{5}\,(-5y - 10z = -20) \quad \text{--- (4)}$$

$$\begin{aligned} -7y - 4z &= 2 \\ y + 2z &= 4. \end{aligned} \quad \text{--- (5)}$$

We now eliminate y in (5), we obtain

$$10z = 30 \quad \text{or} \quad \boxed{z = 3} \quad \text{--- (6)}$$

Substituting this value of z into the first equation of (5), we find that $\boxed{y = -2}$. Then substituting these values of y and z into the first equation of (*), we find that $\boxed{x = 1}$.

We observe further that our elimination procedure has actually produced the linear system

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ z &= 3 \end{aligned} \qquad \cdots (7)$$

obtained by using the first equation of (*), second equation of (5), and (6). The importance of this procedure is that, although the linear systems (7) and (*) are equivalent, (7) has the advantage that it is easier to solve.

Example 6: Consider the linear system

$$\begin{aligned} x + 2y - 3z &= -4 \\ 2x + y - 3z &= 4. \end{aligned} \quad \cdots (18)$$

Find the solution of the linear system (18).

Solution: We eliminate a variable x,

$$-2(x + 2y - 3z) = -2 \cdot -4$$

$$2x + y - 3z = 4 \quad \cdots (19)$$

$$\begin{aligned} -2x - 4y + 6z &= 8 \\ 2x + y - 3z &= 4, \end{aligned} \quad \cdots (20)$$

we obtain

$$-3y + 3z = 12. \quad \cdots (21)$$

We must now solve (21). A solution is

$$y = z - 4,$$

where z can be any real number. Then from the first equation of (18),

$$\begin{aligned} x &= -4 - 2y + 3z \\ &= -4 - 2(z - 4) + 3z \\ &= z + 4. \end{aligned}$$

Thus a solution to the linear system (18) is

$$x = z + 4$$
$$y = z - 4$$
$$z = \text{any real number.}$$

This means that the linear system (18) has infinitely many solutions. Every time we assign a value to z we obtain another solution to (18). Thus, if $z = 1$, then

$$x = 5, \quad y = -3, \quad \text{and} \quad z = 1$$

is a solution, while if $z = -2$, then

$$x = 2, \quad y = -6, \quad \text{and} \quad z = -2$$

is another solution.

Remark 6: The linear system may has no solution such in Example 4, unique solution such in Example 5, and infinitely many solutions such in Example 6.

* Consider a linear system of two equations in the unknowns x and y:

$$\begin{aligned} a_1 x + a_2 y &= c_1 \\ b_1 x + b_2 y &= c_2. \end{aligned} \quad \cdots \quad (22)$$

The graph of each of these equations is a straight line, which we denote by ℓ_1 and ℓ_2, respectively. If $x = s_1$, $y = s_2$ is a solution to the linear system (22), then the point (s_1, s_2) lies on both lines ℓ_1 and ℓ_2. Conversely, if the point (s_1, s_2) lies on both lines ℓ_1 and ℓ_2, then $x = s_1$, $y = s_2$ is a solution to the linear system (22).

Thus we are led geometrically to the same three possibilities mentioned previously. See Figure 1.1.

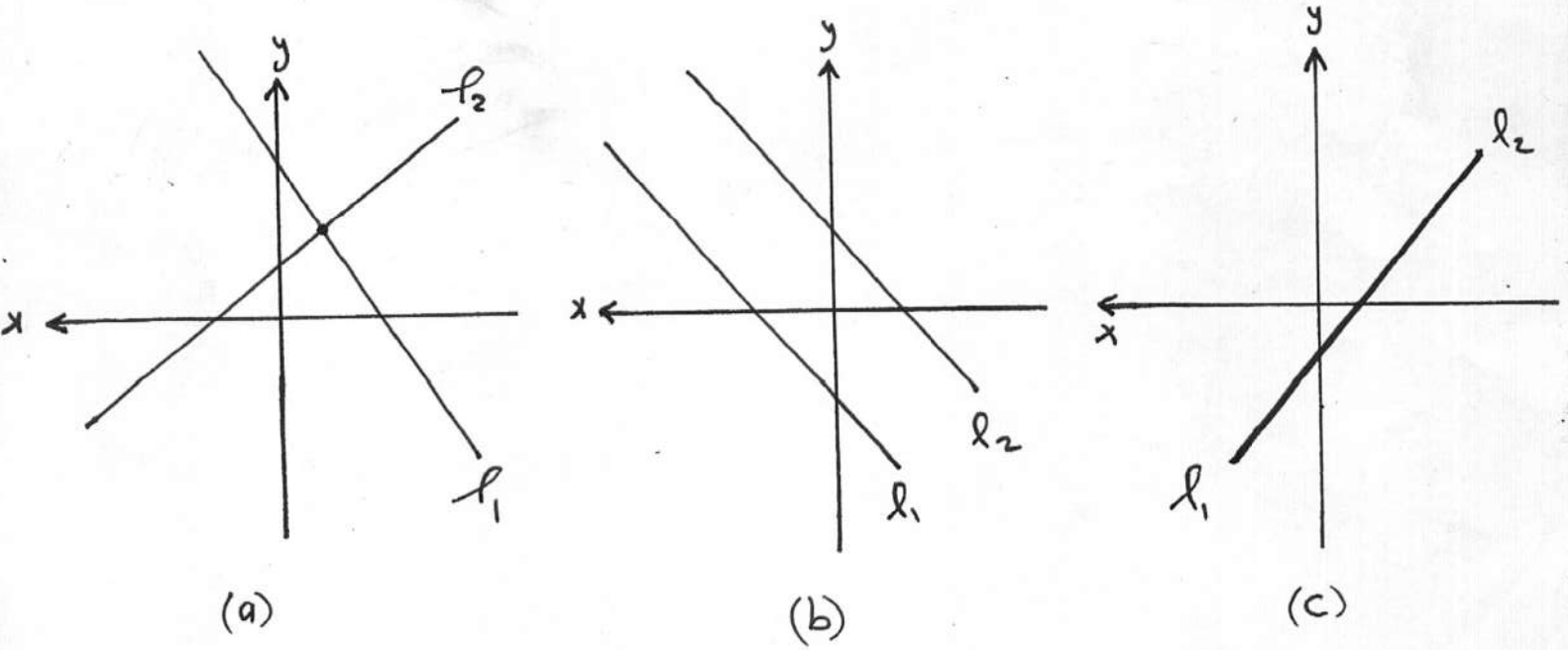


Figure 1.1

Key Terms:

Linear equation, Solution of a linear equation, Linear System, Unknowns, Inconsistent system, Consistent system, Homogeneous system, Trivial solution, Nontrivial solution, Equivalent system, Unique solution, No solution, Infinitely many solutions, Method of elimination.

Exercises:

Q.1 Solve each given linear system by the method of elimination.

a) $x + 2y = 8$
$3x - 4y = 4$

b) $3x + 2y + z = 2$
$4x + 2y + 2z = 8$
$x - y + z = 4$

c) $x + y - 2z = 5$
$2x + 3y + 4z = 1$

Q.2 Given the linear system

$$3x + 4y = s$$
$$6x + 8y = t,$$

a) Determine particular values for s and t so that the system is consistent.

b) Determine particular values for s and t so that the system is inconsistent.

c) What relationship between the values of s and t will gurantee that the system is consistent?

1.2 Matrices

Definition 1: An $m \times n$ matrix A is a rectangular array of mn real or complex numbers arranged in m horizontal rows and n vertical columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & a_{ij} & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} & \end{bmatrix} \leftarrow i^{th} \text{ row} \qquad \cdots (1)$$

$\uparrow$ j^{th} column

* The ith row of A is $[a_{i1} \; a_{i2} \; \cdots \; a_{in}]$ $(1 \leq i \leq m)$.

the jth column of A is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ $(1 \leq j \leq n)$.

Remarks 2: 1) We shall say that A is m by n (written as $m \times n$).

2) If $m = n$, we say that A is a square matrix of order n.

3) The numbers $a_{11}, a_{22}, \cdots, a_{nn}$ form the main diagonal of A.

4) We refer to the number a_{ij}, which is in the ith row and j^{th} column of A, as the i,j^{th} element of A, or the (i,j) entry of A.

5) We often write (1) as $A = [a_{ij}]$.

Example 3: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1+i & 4i \\ 2-3i & -3 \end{bmatrix}$,

$C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$, $E = [3]$, $F = [-1 \quad 0 \quad 2]$.

Then A is a 2×3 matrix with $a_{12} = 2$, $a_{13} = 3$, $a_{22} = 0$ and $a_{23} = 1$;

B is a 2×2 matrix with $b_{11} = 1+i$, $b_{12} = 4i$, $b_{21} = 2-3i$, and $b_{22} = -3$;

C is a 3×1 matrix with $c_{11} = 1$, $c_{21} = -1$, and $c_{31} = 2$;

D is a 3×3 matrix with $d_{11} =$, ... (H.W);

E is a 1×1 matrix with $e_{11} = 3$;

F is a 1×3 matrix with $f_{11} = -1$, $f_{12} = 0$, $f_{13} = 2$.

* In D, the elements $d_{11} = 1$, $d_{22} = 0$, and $d_{33} = 2$ form the main diagonal.

Remark 4: An $n \times 1$ matrix is also called an n-vector.

Example 5: $u = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ is a 4-vector and $v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ is a 3-vector.

Remark 6: The n-vector all of whose entries are zero is denoted by 0.

Remarks 7: ① Observe that if A is an $n \times n$ matrix, then the rows of A are $1 \times n$ matrices and the columns of A are $n \times 1$ matrices.

② The set of all n-vectors with real entries is denoted by R^n. Similarly, the set of all n-vectors with complex entries is denoted by C^n.

Definition 8: Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if they agree entry by entry, that is, if $a_{ij} = b_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Example 9: The matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{bmatrix}$$

are equal if and only if $w = -1$, $x = -3$, $y = 0$, and $z = 5$.

Definition 10: If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then the sum $A + B$ is an $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij},$$

$i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Thus, to obtain the sum of A and B, we merely add corresponding entries.

Example 11: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -4 \end{bmatrix}$.

Then

$$A + B = \begin{bmatrix} 1+0 & -2+2 & 3+1 \\ 2+1 & -1+3 & 4+(-4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & 0 \end{bmatrix}.$$

Remarks 12: ① If X is an n-vector, then it is easy to show that $X + 0 = X$, where 0 is the n-vector all of whose entries are zero.

② It should be noted that the sum of the matrices A and B is defined only when A and B have the same number of rows and the same number of columns, that is, only when A and B are of the same size.

The basic properties of matrix addition are considered in the next section and are similar to those satisfied by the real numbers.

Definition 13: If $A = [a_{ij}]$ is an $m \times n$ matrix and r is a real number, then the scalar multiple of A by r, rA, is the $m \times n$ matrix $C = [c_{ij}]$, where $c_{ij} = ra_{ij}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$; that is, the matrix C is obtained by multiplying each entry of A by r.

Example 14: We have

$$-2\begin{bmatrix} 4 & -2 & -3 \\ 7 & -3 & 2 \end{bmatrix} = \begin{bmatrix} (-2)(4) & (-2)(-2) & (-2)(-3) \\ (-2)(7) & (-2)(-3) & (-2)(2) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 4 & 6 \\ -14 & 6 & -4 \end{bmatrix}.$$

Remark 15: If A and B are $m \times n$ matrices, we write $A + (-1)B$ as $A - B$ and call this the difference between A and B.

Example 16: Let $A = \begin{bmatrix} 2 & 3 & -5 \\ 4 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 5 & -2 \end{bmatrix}$.

Then

$$A - B = \begin{bmatrix} 2-2 & 3+1 & -5-3 \\ 4-3 & 2-5 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -8 \\ 1 & -3 & 3 \end{bmatrix}.$$

Remark 17 : We shall use the summation notation, and we now review this useful and notation. By $\sum_{j=1}^{n} a_j$ we mean

$$a_1 + a_2 + \cdots + a_n.$$

The letter j is called the index of summation ; it is a dummy variable that can be replaced by another letter. Hence we can write

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k.$$

Thus

$$\sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4.$$

The summation notation satisfies the following properties :

1) $\sum_{i=1}^{n} (r_i + s_i)\, a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i .$

2) $\sum_{i=1}^{n} c\,(r_i a_i) = c \sum_{i=1}^{n} r_i a_i .$

3) $\sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right).$

Remark 18 : If $A_1, A_2, \cdots, A_k$ are $m \times n$ matrices and $c_1, c_2, \cdots, c_k$ are real numbers, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_k A_k \quad \cdots\cdots (2)$$

is called a linear combination of $A_1, A_2, \cdots, A_k$, and $c_1, c_2, \cdots, c_k$ are called coefficients.

The linear combination in Equation (2) can also be expressed in summation notation as

$$\sum_{i=1}^{k} c_i A_i = c_1 A_1 + c_2 A_2 + \cdots + c_k A_k .$$

Example 19: The following are linear combinations of matrices:

1) $$3\begin{bmatrix} 0 & -3 & 5 \\ 2 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & -10 & \frac{27}{2} \\ 3 & 8 & \frac{21}{2} \\ \frac{7}{2} & -5 & -\frac{21}{2} \end{bmatrix}$$

2) $$2\begin{bmatrix} 3 & -2 \end{bmatrix} - 3\begin{bmatrix} 5 & 0 \end{bmatrix} + 4\begin{bmatrix} -2 & 5 \end{bmatrix} = \begin{bmatrix} -17 & 16 \end{bmatrix}$$

3) $$-0.5\begin{bmatrix} 1 \\ -4 \\ -6 \end{bmatrix} + 0.4\begin{bmatrix} 0.1 \\ -4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} -0.46 \\ 0.4 \\ 3.08 \end{bmatrix}$$

Definition 20: If $A = [a_{ij}]$ is an $m \times n$ matrix, then the transpose of A, $A^T = [a^T_{ij}]$, is the $n \times m$ matrix defined by

$$a^T_{ij} = a_{ji} .$$

Thus the transpose of A is obtained from A by interchanging the rows and columns of A.

Example 21: Let $A = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & -1 & 2 \\ 0 & 4 & 3 \end{bmatrix}$,

$C = \begin{bmatrix} 5 & 4 \\ -3 & 2 \\ 2 & -3 \end{bmatrix}$, $D = \begin{bmatrix} 3 & -5 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

Then

$$A^T = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -2 \end{bmatrix}, \quad B^T = \begin{bmatrix} 6 & 3 & 0 \\ 2 & -1 & 4 \\ -4 & 2 & 3 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 5 & -3 & 2 \\ 4 & 2 & -3 \end{bmatrix}, \quad D^T = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}, \text{ and } E^T = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}.$$

Key Terms:

Matrix
Rows
Columns
Size of a matrix
Square matrix
Main diagonal
Element or entry of a matrix
Equal matrices
n - vector
R^n, C^n
0, zero vector
Matrix addition
Scalar multiple
Difference of matrix
Summation notation
Index of summation
Linear combination
Coefficients
Transpose

Exercise:

Q.1 If $\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix}$.

Find a, b, c, and d.

Q.2 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$,

$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$, $E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$,

$F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}$, and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

1) Compute the indicated linear combination

(i) $C + E$ (ii) $2C - 3E$ (iii) $-3C - 5O$

2) Compute the following:

(i) A^T and $(A^T)^T$ (ii) $2A^T + B$ (iii) $(2D + 3F)^T$.

(iv) $(C + E + F^T)^T$

1.3 Matrix Multiplication:

In this section we introduce the operation of matrix multiplication. Unlike matrix addition, matrix multiplication has some properties that distinguish it from multiplication of real numbers.

Definition 1: The dot product, or inner product, of the n-vectors in R^n

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$ is defined as

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{j=1}^{n} a_j b_j .$$

Example 2: The dot product of

$$u = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$ is

$$u \cdot v = (1)(2) + (-2)(3) + (3)(-2) + (4)(1) = -6.$$

Example 3: Let $a = \begin{bmatrix} x \\ 2 \\ 3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$. If $a.b = -4$, find x.

We have

$$a.b = 4x + 2 + 6 = -4$$

$$\longrightarrow \quad 4x + 8 = -4 \longrightarrow 4x = -12 \longrightarrow x = -3.$$

Definition 4: If $A = [a_{ij}]$ is an $m \times p$ matrix and $B = [b_{ij}]$ is an $p \times n$ matrix, then the product of A and B denoted AB, is the $m \times n$ matrix $C = [c_{ij}]$, defined by

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj} \quad \cdots (1)$$

$$= \sum_{k=1}^{p} a_{ik} b_{kj} \quad (1 \leq i \leq m, 1 \leq j \leq n).$$

Example 5: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}$.

Then

$$AB = \begin{bmatrix} (1)(-2) + (2)(4) + (-1)(2) & (1)(5) + (2)(-3) + (-1)(1) \\ (3)(-2) + (1)(4) + (4)(2) & (3)(5) + (1)(-3) + (4)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}$$

Remark 6:

$$\underset{m \times p}{A} \quad \underset{p \times n}{B} = \underset{m \times n}{AB}$$

the same

Size of AB

Example 7: Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}.$$

Compute the (3,2) entry of AB.

If $AB = C$, then the (3,2) entry of AB is c_{32}, we now have

$$c_{ij} = \sum_{k=1}^{3} a_{ik} b_{kj}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j}.$$

Hence $c_{32} = a_{31} b_{12} + a_{32} b_{22} + a_{33} b_{32}$

$$= (0)(4) + (1)(-1) + (-2)(2)$$

$$= 0 - 1 - 4 = -5$$

Example 8: Let $A = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix}$.

If $AB = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$, find x and y.

We have $AB = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix} = \begin{bmatrix} 2 + 4x + 3y \\ 4 - 4 + y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$.

Then $2 + 4x + 3y = 12$ and $y = 6$,

So $x = -2$ and $y = 6$.

Remark 9: The product AB is defined only when the number of columns of A is the same as the number of rows of B. Thus, if A is an $m \times p$ matrix and B is a $p \times n$ matrix, then AB is an $m \times n$ matrix. What about BA? Four different situations may occur:

1. BA may not be defined; this will take place if $n \neq m$.
2. If BA is defined, which means that $m = n$, then BA is $p \times p$ while AB is $m \times m$; thus, if $m \neq p$, AB and BA are of different sizes.
3. If AB and BA are both of the same size, they may be equal.
4. If AB and BA are both of the same size, they may be unequal.

Example 10: If A is a 2×3 matrix and B is a 3×4 matrix, then AB is a 2×4 matrix while BA is undefined.

Example 11: Let A be 2×3 and B be 3×2. Then AB is 2×2 while BA is 3×3.

Example 12: Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

Then

$$AB = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix} \text{ while } BA = \begin{bmatrix} 1 & 7 \\ -1 & 3 \end{bmatrix}.$$

Thus $AB \neq BA$.

Remark 13: If u and v are n-vectors ($n \times 1$ matrices), then it is easy to show by matrix multiplication that

$$u \cdot v = u^T v.$$

Remark 14: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ be an $m \times n$ matrix and let

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$ be an n-vector, that is, an $n \times 1$ matrix. Since A is $m \times n$ and c is $n \times 1$, the matrix product Ac is the $m \times 1$ matrix

$$Ac = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n \\ \\ a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n \end{bmatrix}.$$

This last expression can be written as

$$c_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Thus the product Ac of an $m \times n$ matrix A and an $n \times 1$ matrix c can be written as a linear combination, where the coefficients are the entries in the matrix c.

Example 15: Let $A = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix}$ and $c = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

Then the product Ac, written as a linear combination of the columns of A, is

$$Ac = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \end{bmatrix}.$$

* Linear System

Consider the linear system of m equations in n unkowns,

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m . \end{aligned} \qquad \cdots (2)$$

Now define the following matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \cdots (3)$$

$$= \begin{bmatrix} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \end{bmatrix}$$

The entries in the product Ax at the end of (3) are merely the left sides of the equations in (2). Hence the linear system (2) can be written in matrix form as

$$Ax = b.$$

The matrix A is called the coefficient matrix of the linear system (2). The augmented matrix of (2) is written as $[A \vdots b]$. The coefficient and augmented matrices play key roles in method for solving linear systems.

Remark 16: If $b_1 = b_2 = \cdots = b_m = 0$ in (2), the linear system in (2) is called a homogeneous system. A homogeneous system can be written as

$$Ax = 0,$$

where A is the coefficient matrix.

Example 17: Consider the linear system

$$\begin{aligned} -2x \qquad\quad + z &= 5 \\ 2x + 3y - 4z &= 7 \\ 3x + 2y + 2z &= 3. \end{aligned}$$

Letting $A = \begin{bmatrix} -2 & 0 & 1 \\ 2 & 3 & -4 \\ 3 & 2 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $b = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$,

we can write the given linear system in matrix form as

$$A\mathbf{x} = b.$$

The coefficient matrix is A, and the augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 5 \\ 2 & 3 & -4 & 7 \\ 3 & 2 & 2 & 3 \end{array}\right].$$

Example 18: The matrix $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 3 & 0 & 2 & 5 \end{array}\right]$

is the augmented matrix of the linear system

$$\begin{aligned} 2x - y + 3z &= 4 \\ 3x \qquad + 2z &= 5. \end{aligned}$$

Key Terms:

Dot product (inner product)

Matrix - vector product

Coefficient matrix

Augmented matrix

Exercises:

Q.1 Compute $a.b$

(i) $a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ (ii) $a = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$,

(iii) Let $a = b = \begin{bmatrix} -3 \\ 2 \\ x \end{bmatrix}$. If $a.b = 17$, find x.

Q.2 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}$.

Compute the following

(i) BA (ii) $(2AB)^T$ (iii) CB.

Q.3 Let $A = [1 \quad 2 \quad -3]$, $B = [-1 \quad 4 \quad 2]$, and $C = [-3 \quad 0 \quad 1]$.

Compute the following:

(i) AB^T (ii) CC^T (iii) $B^T C A A^T$.

Q.4 Find a value of r and a value of s so that $AB^T = 0$, where $A = [1 \quad r \quad 1]$ and $B = [-2 \quad 2 \quad s]$.

Q.5 Consider the following system:

$$\begin{aligned} 2x_1 + 3x_2 - 3x_3 + x_4 + x_5 &= 7 \\ 3x_1 + 2x_3 + 3x_5 &= -2 \\ 2x_1 + 3x_2 - 4x_4 &= 3 \\ x_3 + x_4 + x_5 &= 5. \end{aligned}$$

(i) Find the coefficient matrix

(ii) Write the linear system in matrix form.

(iii) Find the augmented matrix.

Q.6 Write the following linear system in matrix form:

$$\begin{aligned} -2x_1 + 3x_2 &= 5 \\ x_1 - 5x_2 &= 4 \end{aligned}$$

1.4 Algebraic Properties of Matrix Operations

Theorem 1: (Properties of Matrix Addition)
Let A, B, and C be $m \times n$ matrices.

a) $A + B = B + A$.

b) $A + (B + C) = (A + B) + C$.

c) There is a unique $m \times n$ matrix O such that

$$A + O = A \qquad \text{--- (4)}$$

for any $m \times n$ matrix A. The matrix O is called the $m \times n$ zero matrix.

d) For each $m \times n$ matrix A, there is a unique $m \times n$ matrix D such that

$$A + D = O. \qquad \text{--- (5)}$$

We shall write D as $-A$, so (5) can be written as

$$A + (-A) = O.$$

The matrix $-A$ is called the negative of A. We also note that $-A$ is $(-1)A$.

Proof:

a) Let $A = [a_{ij}]$, $B = [b_{ij}]$,

$A + B = C = [c_{ij}]$, and $B + A = D = [d_{ij}]$.

We must show that $c_{ij} = d_{ij}$ for all i, j. Now

$c_{ij} = a_{ij} + b_{ij}$ and $d_{ij} = b_{ij} + a_{ij}$ for all i, j.

Since a_{ij} and b_{ij} are real numbers, we have

$$a_{ij} + b_{ij} = b_{ij} + a_{ij},$$

which implies that $c_{ij} = d_{ij}$ for all i, j.

B) H.W

c) Let $U = [u_{ij}]$. Then $A + U = A$ if and only if $a_{ij} + u_{ij} = a_{ij}$, which holds if and only if $u_{ij} = 0$. Thus U is the $m \times n$ matrix all of whose entries are zero. U is denoted by O.

D) H.W

Example 2: If $A = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then

$$\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4+0 & -1+0 \\ 2+0 & 3+0 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}.$$

Example 3: If $A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & 4 & 3 \end{bmatrix}$, then $-A = \begin{bmatrix} -1 & -3 & 2 \\ 2 & -4 & -3 \end{bmatrix}$

$$A - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

* Properties of Matrix Multiplication:

a) If A, B, and C are matrices of the appropriate sizes, then

$$A(BC) = (AB)C.$$

b) If A, B, and C are matrices of the appropriate sizes, then

$$(A+B)C = AC + BC.$$

c) If A, B, and C are matrices of the appropriate sizes, then

$$C(A+B) = CA + CB.$$

Example 4: Let $A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 3 & 0 & -1 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}$. Then

$$A(BC) = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 7 \\ 8 & -4 & 6 \\ 9 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}$$

and

$$(AB)C = \begin{bmatrix} 19 & -1 & 6 & 13 \\ 16 & -8 & -8 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}.$$

* Properties of Scalar Multiplication:

If r and s are real numbers and A and B are matricies of the appropriate sizes, then

a) $r(sA) = (rs)A$.

b) $(r+s)A = rA + sA$.

c) $r(A+B) = rA + rB$.

d) $A(rB) = r(AB) = (rA)B$.

Example 5: Let $A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$.

Then

$$2(3A) = 2\begin{bmatrix} 12 & 6 & 9 \\ 6 & -9 & 12 \end{bmatrix} = \begin{bmatrix} 24 & 12 & 18 \\ 12 & -18 & 24 \end{bmatrix} = 6A.$$

We also have

$$A(2B) = \begin{bmatrix} 4 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} 6 & -4 & 2 \\ 4 & 0 & -2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 32 & -10 & 16 \\ 0 & 0 & 26 \end{bmatrix}$$

$$= 2(AB).$$

Theorem 6: If r is a scalar and A and B are matrices of the appropriate sizes, then

a) $(A^T)^T = A$.

b) $(A+B)^T = A^T + B^T$.

c) $(AB)^T = B^T A^T$.

d) $(rA)^T = rA^T$

Proof:

a) Let $A = [a_{ij}] \longrightarrow A^T = [a_{ij}]^T = [a_{ji}]$.

Hence $(A^T)^T = [a_{ji}]^T = [a_{ij}] = A$.

b) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ and let $C = [c_{ij}]$, we have

$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [c_{ij}] = C$.

Then $(A+B)^T = C^T = [c_{ji}]$. On the other hand,

$A^T = [a_{ji}]$ and $B^T = [b_{ji}]$, then

$A^T + B^T = [a_{ji}] + [b_{ji}] = [a_{ji} + b_{ji}] = [c_{ji}] = C^T$.

So $(A+B)^T = A^T + B^T = C^T$.

c) Let $A = [a_{ij}]$ and $B = [b_{ij}]$; let $AB = C = [c_{ij}]$.

We must prove that c^T_{ij} is the (i,j) entry in $B^T A^T$. Now

$$c^T_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk}\, b_{ki} = \sum_{k=1}^{n} a^T_{kj}\, b^T_{ik}$$

$$= \sum_{k=1}^{n} b^T_{ik}\, a^T_{kj} = \text{the } (i,j) \text{ entry in } B^T A^T.$$

Example 7: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$.

Then

$$A^T = \begin{bmatrix} 1 & -2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 3 & 3 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}. \text{ Also,}$$

$$A+B = \begin{bmatrix} 4 & 1 & 5 \\ 1 & 2 & 0 \end{bmatrix} \text{ and } (A+B)^T = \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 5 & 0 \end{bmatrix}.$$

Now

$$A^T + B^T = \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 5 & 0 \end{bmatrix} = (A+B)^T.$$

Example 8: Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}$.

Then

$$AB = \begin{bmatrix} 12 & 5 \\ 7 & -3 \end{bmatrix} \text{ and } (AB)^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}.$$

On the other hand,

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix}.$$

Then

$$B^T A^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix} = (AB)^T.$$

Remark 9: We also note two other peculiarities of matrix multiplication. If a and b are real numbers, then $ab = 0$ can hold only if a or b is zero. However, this is not true for matrices.

Example 10: If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$,

then neither A nor B is the zero matrix, but $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Remark 11: If a, b, and c are real numbers for which $ab = ac$ and $a \neq 0$, it follows that $b = c$. That is, we can cancel out the nonzero factor a. However, the cancellation law does not hold for matrices, as the following example shows.

Example 12: If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$,

then

$$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix}, \text{ but } B \neq C.$$

Remark 13: We summarize some of the differences between matrix multiplication and the multiplication of real numbers as follows: For matrices A, B, and C of the appropriate sizes,

1) AB need not equal BA.
2) AB may be the zero matrix with $A \neq 0$ and $B \neq 0$.
3) AB may equal AC with $B \neq C$.

Key Terms:

Properties of matrix addition
Zero matrix
Properties of matrix multiplication
Properties of scalar multiplication
Properties of transpose

Exercise:

Q.1 Determine a scalar r such that $Ax = rx$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Q.2 Let $A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}$, and $C = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$.

1) Compute $(AB^T)C$.

2) Compute B^TC and multiply the result by A on the right. (Hint: B^TC is 1×1).

Q.3 Determine a constant k such that $(kA)^T(kA) = 1$, where

$$A = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.$$

Q.4 Let $A = \begin{bmatrix} 6 & -1 & 1 \\ 0 & 13 & -16 \\ 0 & 8 & -11 \end{bmatrix}$ and $x = \begin{bmatrix} 10.5 \\ 21.0 \\ 10.5 \end{bmatrix}$.

1) Determine a scalar r such that $Ax = rx$.

2) Is it true that $A^Tx = rx$ for the value r determined in part (1)?

1.5 Special Types of Matrices:

We have already introduced one special type of matrix O, the matrix all of whose entries are zero. We now consider several other types of matrices whose structures are rather specialized and for which it will be convenient to have special names.

An $n \times n$ matrix $A = [a_{ij}]$ is called a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. Note that O is a diagonal matrix.

A scalar matrix is a diagonal matrix whose diagonal elements are equal. The scalar matrix $I_n = [d_{ij}]$, where $d_{ii} = 1$ and $d_{ij} = 0$ for $i \neq j$, is called the $n \times n$ identity matrix.

Example 1: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Then A, B, and I_3 are diagonal matrices; B and I_3 are scalar matrices, and I_3 is the 3×3 identity matrix.

Remark 2: If A is any $m \times n$ matrix, then

$$A I_n = A \text{ and } I_m A = A.$$

Also, if A is a scalar matrix, then $A = r I_n$ for some scalar r.

Remark 3: Suppose that A is a square matrix. We now define the powers of a matrix, for a positive integer p, by

$$A^p = \underbrace{A \cdot A \cdots\cdots A}_{p \text{ factors}}.$$

If A is $n \times n$, we also define

$$A^0 = I_n.$$

For nonnegative integers p and q, the familiar laws of exponents for the real numbers can also be proved for matrix multiplication of a square matrix A:

$$A^p A^q = A^{p+q} \quad \text{and} \quad (A^p)^q = A^{pq}.$$

It should also be noted that the rule

$$(AB)^p = A^p B^p$$

does not hold for square matrix unless $AB = BA$.

An $n \times n$ matrix $A = [a_{ij}]$ is called upper triangular if $a_{ij} = 0$ for $i > j$. It is called lower triangular if $a_{ij} = 0$ for $i < j$. A diagonal matrix is both upper and lower triangular.

Example 4: The matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

is upper triangular, and

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 5 & 2 \end{bmatrix}$$

is lower triangular.

Definition 5: A matrix A with real entries is called symmetric if

$$A^T = A.$$

Definition 6: A matrix A with real entries is called skew symmetric if

$$A^T = -A.$$

Example 7: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is a symmetric matrix.

Example 8: $B = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$ is a skew symmetric matrix.

* Nonsingular Matrices:

Definition 9: An $n \times n$ matrix A is called nonsingular, or invertible, if there exists an $n \times n$ matrix B such that $AB = BA = I_n$; such a B is called an inverse of A. Otherwise, A is called singular, or noninvertible.

Example 10: Let $\begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} = A$ and $B = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$.

Since $AB = BA = I_2$, we conclude that B is an inverse of A.

Theorem 11: The inverse of a matrix, if it exists, is unique.

Proof: Let B and C be inverses of A. Then

$$AB = BA = I_n \quad \text{and} \quad AC = CA = I_n.$$

We then have $B = BI_n = B(AC) = (BA)C = I_nC = C$, which proves that the inverse of a matrix is unique.

Remark 12: Because of this uniqueness, we write the inverse of a nonsingular matrix A as A^{-1}. Thus

$$AA^{-1} = A^{-1}A = I_n.$$

Example 13: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. If A^{-1} exists, let

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we must have

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

So that

$$\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equating corresponding entries of these two matrices, we obtain the linear systems

$$\begin{aligned} a + 2c &= 1 \\ 3a + 4c &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} b + 2d &= 0 \\ 3b + 4d &= 1. \end{aligned}$$

The solutions are $a = -2$, $c = \frac{3}{2}$, $b = 1$, and $d = -\frac{1}{2}$. Moreover, Since the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix},$$

also satisfies the property that

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

We conclude that A is nonsingular and that

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Theorem 14: If A and B are both nonsingular $n \times n$ matrices, then AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: We have $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = (AI_n)A^{-1}$

$= AA^{-1} = I_n$.

Similarly, $(B^{-1}A^{-1})(AB) = I_n$. Therefore AB is nonsingular. Since the inverse of a matrix is unique, we conclude that $(AB)^{-1} = B^{-1}A^{-1}$.

Corollary 15: If $A_1, A_2, \ldots, A_r$ are $n \times n$ nonsingular matrices, then $A_1 A_2 \cdots A_r$ is nonsingular and $(A_1 A_2 \cdots A_r)^{-1} = A_r^{-1} A_{r-1}^{-1} \cdots A_1^{-1}$.

Theorem 16: If A is a nonsingular matrix, then A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

Theorem 17: If A is a nonsingular matrix, then A^T is nonsingular and $(A^{-1})^T = (A^T)^{-1}$.

Proof: We have $AA^{-1} = I_n$. Taking transposes of both sides, we get

$$(A^{-1})^T A^T = I_n^T = I_n.$$

Taking transposes of both sides of the equation $A^{-1}A = I_n$, we find, similarly, that

$$(A^T)(A^{-1})^T = I_n.$$

These equations imply that $(A^{-1})^T = (A^T)^{-1}$.

Example 18: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then from Example 13

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad (A^{-1})^T = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}.$$

Also,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad (A^T)^{-1} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}.$$

Remark 19: Suppose that A is nonsingular. Then $AB = AC$ implies that $B = C$, and $AB = 0$ implies that $B = 0$.

* Linear Systems and Inrerses:

If A is an $n \times n$ matrix, then the linear system $Ax = b$ is a system of n equations in n unknowns. Suppose that A is nonsingular. Then A^{-1} exists, and we can multiply $Ax = b$ by A^{-1} on the left on both sides, yielding

$$A^{-1}(Ax) = A^{-1}b$$
$$(A^{-1}A)x = A^{-1}b$$
$$I_n x = A^{-1}b$$
$$x = A^{-1}b. \quad \text{---} \quad (1)$$

Moreover, $x = A^{-1}b$ is clearly a solution to the given linear system. Thus, if A is nonsingular, we have a unique solution.

Remark 20: In equation (1), if $b = 0$, then the unique solution to the homogeneous system $Ax = 0$ is $x = 0$.

Remark 21: If A is a nonsingular $n \times n$ matrix, Equation (1) implies that if the linear system $Ax = b$ needs to be solved repeatedly for different b's, we need compute A^{-1} only once; then whenever we change b, we find the corresponding solution x by forming $A^{-1}b$.

Example 22: Suppose that A is the matrix of Example 13 so that

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

If $b = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$, then the solution to the linear system $Ax = b$ is

$$x = A^{-1}b = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} -10 \\ 9 \end{bmatrix}.$$

On the other hand, if

$$b = \begin{bmatrix} 10 \\ 20 \end{bmatrix},$$

then

$$x = A^{-1} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Key Terms

Diagonal Matrix

Identity Matrix

Powers of a Matrix

Upper Triangular Matrix

Lower Triangular Matrix

Symmetric Matrix

Skew symmetric Matrix

Nonsingular (invertible) Matrix

Inverse

Singular (noninvertible) Matrix

Properties of Nonsingular Matrices

Linear System with Nonsingular Coefficient Matrix

Exercises:

Q.1 Prove: If A and B are $n \times n$ diagonal matrices, then $AB = BA$.

Q.2 Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -3 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$. Verify that $A+B$ and AB are upper triangular.

Q.3 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$. Compute each of the following:

a) A^2 b) $(AB)^2$.

Q.4 If $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find D^{-1}.

Q.5 If A is a nonsingular matrix whose inverse is $\begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$, find A.

Q.6 Suppose that $A^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. Solve the linear system $Ax = b$ of $b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

1.6 Matrix Transformations:

In Section 1.2 we introduced the notation R^n for the set of all n-vectors with real entries. Thus R^2 denotes the set of all 2-vectors and R^3 denotes the set of all 3-vectors.

The vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ in R^2 is represented by the directed line segment shown in Figure 1. The vector

$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in R^3 is represented by the directed line segment shown in Figure 2.

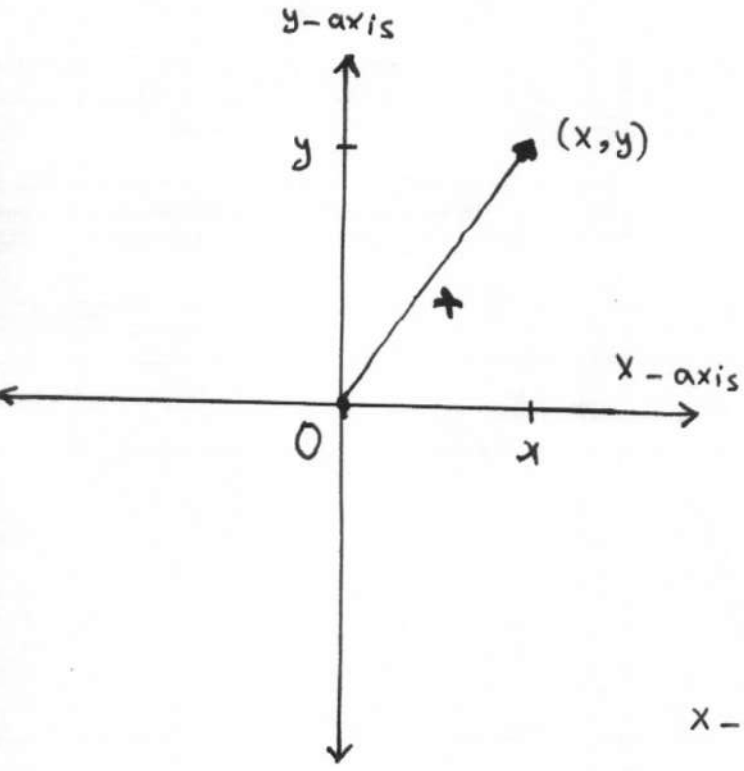


Figure 1

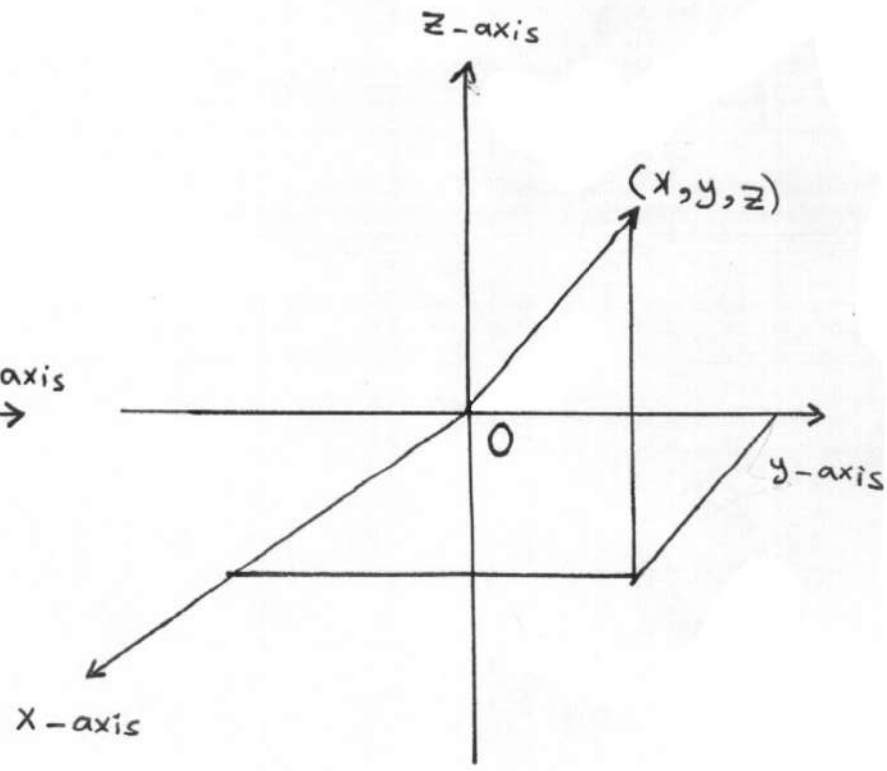


Figure 2

Example 1: Figur 3 shows geometric representations of the 2-vectors

$$u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \text{ and } u_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

in a 2-dimensional rectangular coordinate system.

Figure 4 shows geometric representations of the 3-vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

in a 3-dimensional rectangular coordinate system.

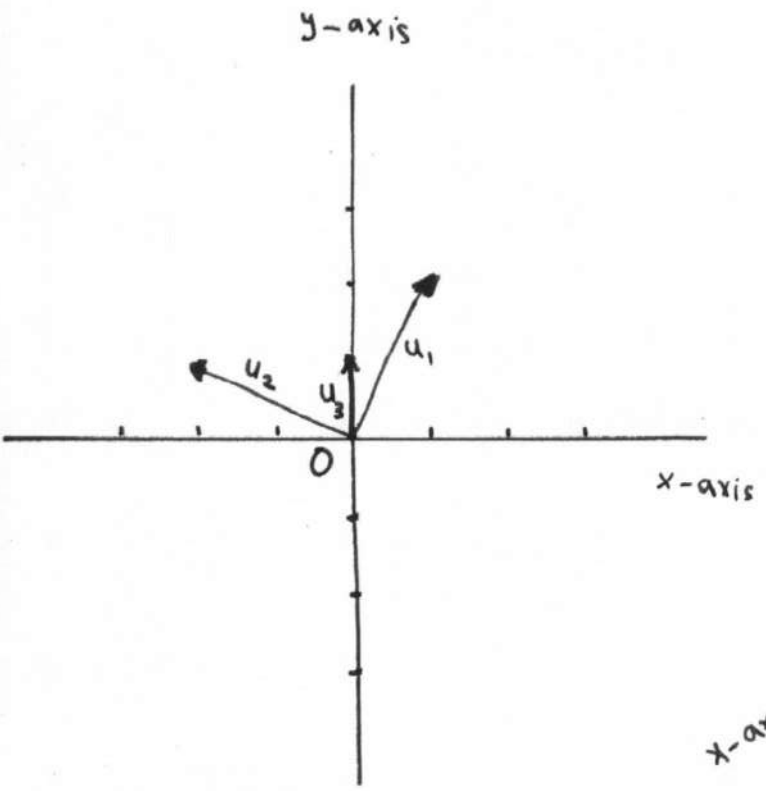


Figure 3

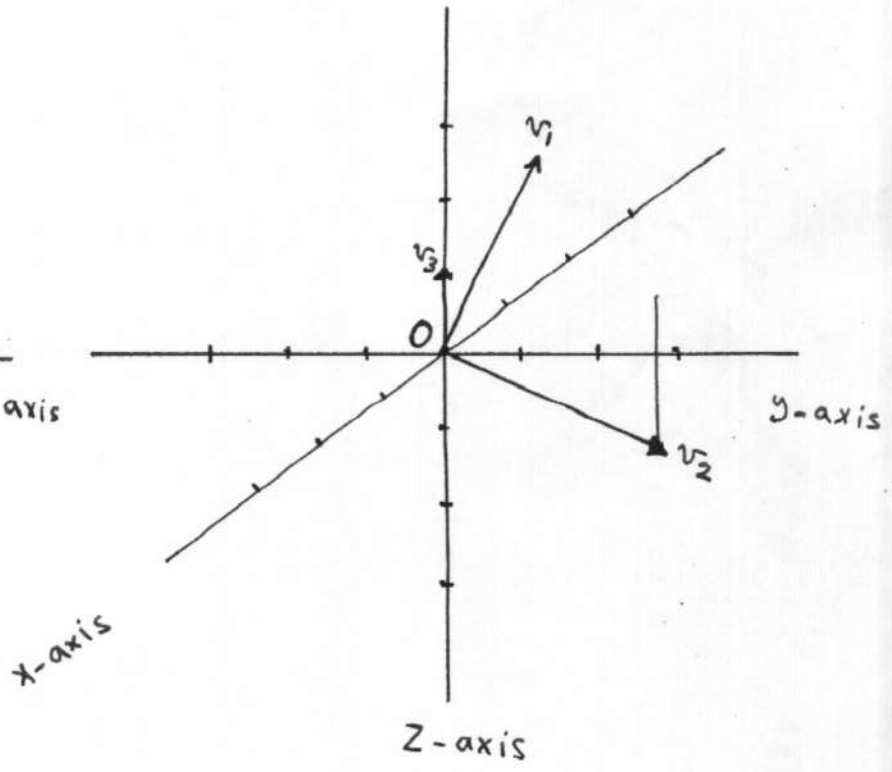


Figure 4

Remark 2: If A is an $m \times n$ matrix and u is an n-vector, then the matrix product Au is an m-vector. A function f mapping R^n into R^m is denoted by $f: R^n \longrightarrow R^m$. A matrix transformation is a function $f: R^n \longrightarrow R^m$ defined by $f(u) = Au$. The vector $f(u)$ in R^m is called the image of u, and the set of all images of the vectors in R^n is called the range of f.

Example 3: Let f be the matrix transformation defined by

$$f(u) = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} u.$$

The image of $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is

$$f(u) = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix},$$

and the image of $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$.

Remark 4: If A is an $m \times n$ matrix and $f: R^n \longrightarrow R^m$ is a matrix transformation mapping R^n into R^m that is defined by $f(u) = Au$, then a vector W in R^m is in the range of f only if we can find a vector v in R^n such that $f(v) = W$.

Example 5: Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ and consider the matrix transformation defined by $f(u) = Au$. Determine if the vector $W = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is in the range of f.

Solution: The question is equivalent to asking whether there is a vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $f(v) = W$. We have

$$Av = \begin{bmatrix} v_1 + 2v_2 \\ -2v_1 + 3v_2 \end{bmatrix} = W = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

or

$$\begin{aligned} v_1 + 2v_2 &= 4 \\ -2v_1 + 3v_2 &= -1. \end{aligned}$$

Solving this linear system of equations by the familiar method of elimination, we get $v_1 = 2$ and $v_2 = 1$. Thus W is in the range of f. In particular, if $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then $f(v) = W$.

Key Terms:

Matrix Transformation

Mapping (function)

Range

Image

Exercises:

Q.1 Sketch u and its image under given matrix transformation f

$f: R^2 \longrightarrow R^2$ defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Q.2 Sketch u and its image under given matrix transformation f

$f: R^3 \longrightarrow R^3$ defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad u = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}.$$

Q.3 Let $f: R^2 \longrightarrow R^3$ for the matrix transformation defined by

$f(u) = Au$, where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Determine whether a vector $W = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is in the range of f.

Chapter 2: Solving Linear Systems:

2.1 Echelon Form of a Matrix.

Definition 1: An $m \times n$ matrix A is said to be in reduced row echelon form if it is satisfies the following properties:

1) All zero rows, if there are any, appear at the bottom of the matrix.
2) The first nonzero entry from the left of a nonzero row is a 1. This entry is called a leading one of its row.
3) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.
4) If a column contains a leading one, then all other entries in that column are zero.

Example 2: The following are matrices in reduced row echelon form, Since they satisfy properties 1), 2), 3), and 4):

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \text{ and}$$

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrices that follow are not in reduced row echelon form. (Why not?)

$$D=\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad E=\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

$$F=\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Definition 3: An elementary row operation on a matrix A is any of the following operations:

1) Type I: Interchange any two rows.
2) Type II: Multiply a row by a nonzero number.
3) Type III: Add a multiple of one row to another.

Remark 4: We now introduce the following notation for elementary row operations on matrices:

* Interchange rows i and j, Type I:

$$r_i \longleftrightarrow r_j$$

* Replace i by k times row i, Type II:

$$kr_i \longrightarrow r_i$$

* Replace row j by k times row i + row j, Type III:

$$kr_i + r_j \longrightarrow r_j$$

Example 5 : Let $A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}$.

Interchanging rows 1 and 3 of A, we obtain

$$B = A_{r_1 \leftrightarrow r_3} = \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Multiplying the third row of A by $\frac{1}{3}$, we obtain

$$C = A_{\frac{1}{3}r_3 \to r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}.$$

Adding (-2) times row 2 of A to row 3 of A, we obtain

$$D = A_{-2r_2 + r_3 \to r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}.$$

Definition 6: Let **B** is an mxn matrix is said to be row equivalent to an mxn matrix **A** if B can be produced by applying a finite sequence of elementary row operations to A.

Example 7: Let $A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -2 & 2 & 3 \end{bmatrix}$.

If we add 2 times row 3 of A to its second row, we obtain

$$B = A_{2r_3 + r_2 \to r_2} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & -3 & 7 & 8 \\ 1 & -2 & 2 & 3 \end{bmatrix}$$

So B is row equivalent to A.

Interchanging rows 2 and 3 of B, we obtain

$$C = B_{r_2 \leftrightarrow r_3} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & -2 & 2 & 3 \\ 4 & -3 & 7 & 8 \end{bmatrix},$$

So C is row equivalent to B.

Example 8: Find a matrix in reduced row echelon form that is row equivalent to the matrix A, where

$$A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

Pivot

Pivot column

Solution: Column 1 is the first (counting from left to right) column in A with a nonzero entry, so column 1 is the pivot column of A. The first (counting from top to bottom) nonzero entry in the pivot column occurs in the third row, so the pivot is $a_{31} = 2$.

We interchange the first and third rows of A, obtaining

$$B = A_{r_1 \leftrightarrow r_3} = \begin{bmatrix} \textcircled{2} & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}.$$

Multiply the first row of B by the reciprocal of the pivot, that is, by $\frac{1}{b_{11}} = \frac{1}{2}$, to obtain

$$C = B_{\frac{1}{2}r_1 \to r_1} = \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}.$$

Add (-2) times the first row of C to the fourth row of C to produce a matrix D in which the only nonzero entry in the pivot column is $d_{11} = 1$;

$$D = C_{-2r_1 + r_4 \to r_4} = \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}.$$

Identify A_1 as the submatrix of D obtained by deleting the first row of D

$$A_1 = \begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & \textcircled{2} & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}.$$

Pivot column

Pivot

Do $(A_1)_{r_1 \leftrightarrow r_2}$ to obtain B_1.

$$B_1 = \begin{bmatrix} \boxed{1 & 1 & -\frac{5}{2} & 1 & 2} \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

. $D_0(B_1)$ $\frac{1}{2}r_1 \to r_1$ to obtain C_1.

$$C_1 = \begin{bmatrix} \boxed{1 & 1 & -\frac{5}{2} & 1 & 2} \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

. $D_0(C_1)$ $2r_1 + r_3 \to r_3$ to obtain D_1.

$$D_1 = \begin{bmatrix} \boxed{1 & 1 & -\frac{5}{2} & 1 & 2} \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}.$$

Deleting the first row of D_1 yields the matrix A_2.

$$A_2 = \begin{bmatrix} 0 & 0 & \textcircled{2} & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}.$$

Pivot column of A_2

Pivot

$D_0(A_2)$ $\frac{1}{2}r_1 \to r_1$ to obtain C_2.

$$C_2 = \begin{bmatrix} \boxed{1 & 1 & -\frac{5}{2} & 1 & 2} \\ \boxed{0 & 1 & \frac{3}{2} & -2 & \frac{1}{2}} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}.$$

Finally, do (C_2) $-2r_1 + r_2 \to r_2$ to obtain D_2.

$$\begin{array}{|ccccc|}\hline 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ \hline \end{array}$$

$$D_2 = \begin{bmatrix} 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we have the matrix

$$H = \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_1 = H \underset{-\frac{3}{2}r_3 + r_2 \to r_2}{} = \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_2 = (J_1) \underset{\frac{5}{2}r_3 + r_1 \to r_1}{} = \begin{bmatrix} 1 & 1 & 0 & \frac{19}{4} & 7 \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$K = (J_2) \underset{-1 \cdot r_2 + r_1 \to r_1}{} = \begin{bmatrix} 1 & 0 & 0 & 9 & \frac{19}{2} \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 9: Let $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$. Find a matrix in reduced row echelon form that is row equivalent to the matrix A.

Solution:

Pivot → $A = \begin{bmatrix} \textcircled{1} & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$

Pivot column ↑

$$B = A_{2r_1 + r_2 \to r_2} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix}$$

$$C = B_{-3r_1 + r_3 \to r_3} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$D = C_{r_2 + r_3 \to r_3} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E = D_{-2r_2 + r_1 \to r_1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus E is now in reduced row echelon form.

Remark 10: An $m \times n$ matrix satisfying properties 1), 2), and (3) is said to be in row echelon form.

Key Terms:

Reduced row echelon form
Leading one
Row echelon form
Row equivalent
Elementary row operation
Pivot column
Pivot

Exercises:

Q.1 Let $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 9 \\ 3 & 2 & 4 \end{bmatrix}$. Find the reduced row echelon form of a matrix A.

Q.2 Let x, y, z, and w be nonzero real numbers. Label each of the following matrices REF if it is in row echelon form, $RREF$ if it is in reduced row echelon form, or N if it is not REF and not $RREF$:

1) $A = \begin{bmatrix} 1 & x & y & 0 \\ 0 & 1 & 0 & z \\ 0 & 0 & w & 1 \end{bmatrix}$ 2) $B = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Q.3 Determine the reduced row echelon form of

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

2.2 Solving Linear Systems:

In this section we use the echelon forms developed in Section 2.1 to more efficiently determine the solution of a linear system compared with the elimination method of Section 1.1. Using the augmented matrix of a linear system together with an echelon form, we develop methods for solving a system of m linear equations in n unknows. These methods take the augmented matrix of the linear system, perform elementary row operations on it, and obtain a new matrix that represents an equivalent linear system (i.e. a system that has the same solutions as the original linear system). The important point is that the latter linear system can be solved more easily.

To see how a linear system whose augmented matrix has a particular form can be readily solved, suppose that

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

represents the augmented matrix of a linear system. Then the solution is quickly found from the corresponding equations

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_2 + x_3 &= 2 \\ x_3 &= -1 \end{aligned}$$

as

$$x_3 = -1 \; ; \; x_2 = 2 - x_3 = 2 + 1 = 3 \; ; \; x_1 = 3 - 2x_2 = 3 - 6 = -3.$$

Remark 1: Let $Ax = b$ and $Cx = d$ be two linear systems, each of m equations in n unknows. If the augmented matrices $[A \mid b]$ and $[C \mid d]$ are row equivalent, then the linear systems are equivalent; that is, they have exactly the same solutions.

Remark 2: If A and C are row equivalent $m \times n$ matrices, then the homogeneous systems $Ax = 0$ and $Cx = 0$ are equivalent.

* We introduce two methods to solve a linear system $Ax = b$.

First method: Gaussian elimination consists of two steps:

Step 1: The transformation of the augmented matrix $[A \mid b]$ to the matrix $[C \mid d]$ in row echelon form using elementary row operations.

Step 2: Solution of the linear system corresponding to the augmented matrix $[C \mid d]$ using back substitution.

Example 3: The linear system

$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x - y + z &= 8 \\ 3x \quad - z &= 3 \end{aligned}$$

has the augmented matrix

$$[A \mid b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right].$$

Transforming this matrix to row echelon form, we obtain

$$[C \mid d] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Using back substitution, we now have

$$z = 3$$
$$y = 2 - z = 2 - 3 = -1$$
$$x = 9 - 2y - 3z = 9 + 2 - 9 = 2;$$

thus the solution is $x = 2$, $y = -1$, $z = 3$, which is unique.

Example 4: Let

$$[C \mid d] = \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & -1 & 7 \\ 0 & 0 & 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 & 2 & 9 \end{array}\right].$$

Then

$$x_4 = 9 - 2x_5$$
$$x_3 = 7 - 2x_4 - 3x_5 = 7 - 2(9 - 2x_5) - 3x_5 = -11 + x_5$$
$$x_2 = 7 - 2x_3 - 3x_4 + x_5 = 2 + 5x_5$$
$$x_1 = 6 - 2x_2 - 3x_3 - 4x_4 - 5x_5 = -1 - 10x_5$$
$$x_5 = \text{any real number.}$$

The system is consistent, and all solutions are of the form

$$x_1 = -1 - 10r$$
$$x_2 = 2 + 5r$$
$$x_3 = -11 + r$$
$$x_4 = 9 - 2r$$
$$x_5 = r, \text{ any real number.}$$

Since r can be assigned any real number, the given linear system has infinitely many solutions.

Example 5: If

$$[C \mid d] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right], \text{ then}$$

$Cx = d$ has no solution, since the last equation is

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1,$$

which can never be satisfied.

Second method: When using the Gauss-Jordan reduction procedure, we transform the augmented matrix $[A \mid b]$ to $[C \mid d]$, which is in reduced row echelon form. This means that we can solve the linear system $Cx = d$ without back substitution. We now solve a linear system both by Gaussian elimination and by Gauss-Jordan reduction.

Example 6: Consider the linear system

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2. \end{aligned}$$

We form the augmented matrix

$$[A \mid b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$[A \mid b]_{-2r_1 + r_2 \to r_2} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$[A \mid b] \begin{matrix} \\ -2r_1 + r_2 \to r_2 \\ -3r_1 + r_3 \to r_3 \end{matrix} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right].$$

$$[A \mid b] \begin{matrix} \\ -2r_1 + r_2 \to r_2 \\ -3r_1 + r_3 \to r_3 \\ -\frac{1}{5} r_3 \to r_3 \end{matrix} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$$[A \mid b] \begin{matrix} \\ -2r_1 + r_2 \to r_2 \\ -3r_1 + r_3 \to r_3 \\ -\frac{1}{5} r_3 \to r_3 \\ r_2 \leftrightarrow r_3 \end{matrix} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right]$$

$$[A \mid b] \begin{matrix} \\ -2r_1 + r_2 \to r_2 \\ -3r_1 + r_3 \to r_3 \\ -\frac{1}{5} r_3 \to r_3 \\ r_2 \leftrightarrow r_3 \\ 7r_2 + r_3 \to r_3 \end{matrix} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right]$$

$$[A \mid b] \underset{\substack{-2r_1 + r_2 \to r_2 \\ -3r_1 + r_3 \to r_3 \\ \frac{1}{5} r_3 \to r_3 \\ r_2 \leftrightarrow r_3 \\ 7r_2 + r_3 \to r_3 \\ \frac{1}{10} r_3 \to r_3}}{=} \left[\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

This matrix is in row echelon form. This means that $z = 3$, and from the second row,

$$y + 2z = 4$$

so that

$$y = 4 - 2(3) = -2.$$

From the first row,

$$x + 2y + 3z = 6,$$

which implies that

$$x = 6 - 2y - 3z = 6 - 2(-2) - 3(3) = 1.$$

Thus $x = 1$, $y = -2$, and $z = 3$ is the solution. This gives the solution by Gaussian elimination.

To solve the given linear system by Gauss-Jordan reduction, we transform last matrix to $[C \mid d]$, which is in reduced row echelon form, by the following steps:

$$[C \mid d] \underset{-2r_3 + r_2 \to r_2}{=} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

$$[C \mid d] \underset{\substack{-2r_3 + r_2 \to r_2 \\ -3r_3 + r_1 \to r_1}}{=} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

$$[C \mid d] \underset{\substack{-2r_3 + r_2 \to r_2 \\ -3r_3 + r_1 \to r_1 \\ -2r_2 + r_1 \to r_1}}{=} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

The solution is $x = 1$, $y = -2$, and $z = 3$, as before.

Key Terms:

Gaussian elimination

Gauss-Jordan reduction

Back substitution

* Homogeneous Systems:

Now we study a homogeneous system $Ax = 0$ of m linear equations in n unknowns.

Example 7: Consider the homogeneous system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Since the augmented matrix is in reduced row echelon form, the solution is seen to be

$$x_1 = -2r$$
$$x_2 = \quad s$$
$$x_3 = -3r$$
$$x_4 = -4r$$
$$x_5 = \quad r,$$

where r and s are any real numbers.

Remark 8: In Example 7 we solved a homogeneous system of $m = 4$ linear equations in $n = 5$ unknows, where $m < n$ and the augmented matrix A was in reduced row echelon form.

Theorem 9: A homogeneous system of m linear equations in n unknowns always has a nontrivial solution if $m < n$, that is, if the number of unknows exceeds the number of equations.

Proof: Let B be a matrix in reduced row echelon form that is row equivalent to A. Then the homogeneous systems $Ax = 0$ and $Bx = 0$ are equivalent. As we have just shown, the system $Bx = 0$ has a nontrivial solution, and therefore the same is true for the system $Ax = 0$.

Remark 10: We shall use this result in the following form:
If A is $m \times n$ and $Ax = 0$ has only the trivial solution, then $m \geq n$.

Example 11: Consider the homogeneous system

$$\begin{aligned} x + y + z + w &= 0 \\ x \qquad\quad + w &= 0 \\ x + 2y + z \qquad &= 0. \end{aligned}$$

The augmented matrix

$$A = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{array}\right]$$

is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right].$$

Hence the solution is $x = -r$, $y = r$, $z = -r$, and $w = r$, any real number.

Exercises :

Q.1 Consider the linear system

$$\begin{aligned} x + y + 2z &= -1 \\ x - 2y + z &= -5 \\ 3x + y + z &= 3. \end{aligned}$$

1) Find all solutions, if any exist, by using the Gaussian elimination method.

2) Find all solutions, if any exist, by using the Gauss-Jordan reduction method.

Q.2 Solve the linear system, with the given augmented matrix, if it is consistent.

1) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right]$ 2) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 5 & 7 & 9 & 0 \end{array}\right]$

Q.3 Find a 2×1 matrix x with entries not all zero such that

$$Ax = 4x, \text{ where } A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}.$$

Hint : Rewrite the matrix equation $Ax = 4x$ as $4x - Ax = (4I_2 - A)x = 0$, and solve the homogeneous linear system.

Q.4 Let $f: R^3 \longrightarrow R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}. \text{ Find } x, y, z \text{ so that}$$

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}.$$

2.3 Elementary Matrices; Finding A^{-1}

In this section we develop a method for finding the inverse of a matrix if it exists. To use this method we do not have to find out first whether A^{-1} exists. This method requires that elementary row operations of types I, II, and III (see Section 2.1) be performed on A. We clarify these notions by starting with the following definition:

Definition 1: An nxn elementary matrix of type I, type II, or type III is a matrix obtained from the identity matrix I_n by performing a single elementary row operation of type I, type II, or type III, respectively.

Example 2: The following are elementary matrices:

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix E_1 is of type I - we interchanged the first and third rows of I_3; E_2 is of type II - we multiplied the second row of I_3 by (-2); E_3 is of type III - we added twice the second row of I_3 to the first row of I_3; and E_4 is of type III - we added three times the first column of I_3 to the third column of I_3.

Remark 3: Let A be an $m \times n$ matrix, and let an elementary row operation of type I, type II, and type III be performed on A to yield matrix B. Let E be the elementary matrix obtained from I_m (I_n) by performing the same elementary row operation as was performed on A. Then $B = EA$ ($B = AE$).

Example 4: Let

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ -1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 2 \end{bmatrix}$$

and let $B = A$ $\xrightarrow{-2r_3 + r_1 \to r_1}$; then

$$B = \begin{bmatrix} -5 & 3 & 0 & -3 \\ -1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 2 \end{bmatrix}.$$

Now let $E = (I_3)$ $\xrightarrow{-2r_3 + r_1 \to r_1}$; then

$$E = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can readily verify that $B = EA$.

Remark 5: If A and B are $m \times n$ matrices, then A is row equivalent to B if and only if there exists elementary matrix E such that $B = EA$ ($B = AE$).

Theorem 6: An elementary matrix E is nonsingular, and its inverse is an elementary matrix of the same type.

Proof:

If E_1 is an elementary matrix of type I then $E_1^{-1} = E_1$. Let E_2 be obtained from I_n by multiplying the ith row of I_n by $c \neq 0$. Let E_2^* be obtained from I_n by multiplying the ith row of I_n by $\frac{1}{c}$. Then $E_2 E_2^* = I_n$. Let E_3 be obtained from I_n by adding c times the ith row of I_n to the jth row of I_n. Let E_3^* be obtained from I_n by adding $-c$ times the ith row of I_n to the jth row of I_n. Then $E_3 E_3^* = I_n$.

Remark 7: Let A be an $n \times n$ matrix and let the homogeneous system $Ax = 0$ have only the trivial solution $x = 0$. Then A is row equivalent to I_n. (That is, the reduced row echelon form of A is I_n.)

Theorem 8: A is nonsingular if and only if A is a product of elementary matrices.

Proof: If A is a product of elementary matrices $E_1, E_2, \ldots, E_k$, then $A = E_1 E_2 \cdots E_k$. Now each elementary matrix is nonsingular, and by (Theorem 14 - Sec. 1.5), the product of nonsingular matrices is nonsingular therefore A is nonsingular.

Conversely, if A is nonsingular, then $Ax = 0$ implies that $A^{-1}(Ax) = A^{-1}0 = 0$ so $I_n x = 0$ or $x = 0$. Thus $Ax = 0$ has only the trivial solution. Remark 7 then implies that A is row equivalent to I_n. This means that there exist

elementary matrices $E_1, E_2, \cdots, E_k$ such that

$$I_n = E_k E_{k-1} \cdots E_2 E_1 A.$$

It then follows that $A = (E_k E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$. Since the inverse of an elementary matrix is an elementary matrix, we have established the result.

Corollary 9: A is nonsingular if and only if A is row equivalent to I_n. (That is, the reduced row echelon form of A is I_n.)

Remark 10: The homogeneous system of n linear equations in n unknowns $Ax = 0$ has a nontrivial solution if and only if A is singular. (That is, the reduced row echelon form of $A \neq I_n$.)

Example 11: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Consider the homogeneous system $Ax = 0$; that is,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The augmented matrix $[A \,\vdots\, b]$ is

$$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right].$$

The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

So a solution is $x = -2r$ and $y = r$,

where r is any real number. Thus the homogeneous system has a nontrivial solution, And A is singular.

Remark 12: We have shown that if the $n \times n$ matrix A is nonsingular, then the system $Ax = b$ has a unique solution for every $n \times 1$ matrix b. Also the converse of this statement is also true.

* Note that at this point we have shown that the following statements are equivalent for $n \times n$ matrix A:

1) A is nonsingular.
2) $Ax = 0$ has only the trivial solution.
3) A is row equivalent to I_n. (The reduced row echelon form of A is I_n.)
4) The linear system $Ax = b$ has a unique solution for every $n \times 1$ matrix b.
5) A is a product of elementary matrices.

* Finding A^{-1}: For a nonsingular matrix A, we transform the partitioned matrix $[A \mid I_n]$ to reduced row echelon form, obtaining $[I_n \mid A^{-1}]$.

Example 13: Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}.$$

Assuming that A is nonsingular, we form

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right].$$

We now perform elementary row operations that transform $[A \mid I_3]$ to $[I_3 \mid A^{-1}]$. We arrange our computations, using elementary row operations as follows:

A	I_3	
$\begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	Apply $-5r_1 + r_3 \to r_3$.
$\begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -4 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ -5 & 0 & 1 \end{matrix}$	Apply $\frac{1}{2}r_2 \to r_2$.
$\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & -4 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ -5 & 0 & 1 \end{matrix}$	Apply $-\frac{1}{4}r_3 \to r_3$.
$\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{matrix}$	Apply $-\frac{3}{2}r_3 + r_2 \to r_2$ and $-1\,r_3 + r_1 \to r_1$.
$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} -\frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{matrix}$	Apply $-1\,r_2 + r_1 \to r_1$.
$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{matrix}$	

Hence
$$A^{-1} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}.$$

We can readily verify that $AA^{-1} = A^{-1}A = I_3$.

Remark 14: An $n \times n$ matrix A is singular if and only if A is row equivalent to a matrix B that has a row of zeros. (That is, the reduced row echelon form of A has a row of zeros.)

* This means that in order to find A^{-1}, we do not have to determine, in advance, whether it exists. We merely start to calculate A^{-1}; if at any point in the computation we find a matrix B that is row equivalent to A and has a row of zeros, then A^{-1} does not exist. That is, we transform the augmented matrix $[A \,|\, I_n]$ to reduced row echelon form, obtaining $[C \,|\, D]$. If $C = I_n$, then $D = A^{-1}$. If $C \neq I_n$, then C has a row of zeros and we conclude that A is singular.

Example 15: Let
$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}.$$

To find A^{-1}, we proceed as follows:

A			I_3			
1	2	-3	1	0	0	Apply $-1r_1 + r_2 \to r_2$.
1	-2	1	0	1	0	
5	-2	-3	0	0	1	
1	2	-3	1	0	0	Apply $-5r_1 + r_3 \to r_3$
0	-4	4	-1	1	0	
5	-2	-3	0	0	1	
1	2	-3	1	0	0	Apply $-3r_2 + r_3 \to r_3$
0	-4	4	-1	1	0	
0	-12	12	-5	0	1	
1	2	-3	1	0	0	
0	-4	4	-1	1	0	
0	0	0	-2	-3	1	

At this point A is row equivalent to

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$

the last matrix under A. Since B has a row of zeros, we stop and conclude that A is a singular matrix.

Remark 16: In section 1.5 (Definition 9) we defined an $n \times n$ matrix B to be the inverse of the $n \times n$ matrix A if $AB = I_n$ and $BA = I_n$. We introduce a fact that one of these equations follows from the other.

Proposition 17: If A and B are $n \times n$ matrices such that $AB = I_n$ then $BA = I_n$. Thus $B = A^{-1}$.

Key Terms:

Elementary Matrix
Inverse Matrix
Nonsingular Matrix
Singular Matrix

Exercises:

Q.1 Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. Find a matrix C in reduced row echelon form that is row equivalent in A. Record the row operations used.

Q.2 Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

Q.3 Which of the given matrices are singular? For the nonsingular ones, find the inverse.

1) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$ 3) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

Q.4 If A is a nonsingular matrix whose inverse is $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$, find A.

Q.5 Find all value(s) of a for which the inverse of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix}$$

exists. What is A^{-1} ?

Q.6 For what values of a does the homogeneous system

$$\begin{aligned} (a-1)x + 2y &= 0 \\ 2x + (a-1)y &= 0 \end{aligned}$$

have a nontrivial solution?

Chapter 3 Determinants

3.1 Definition

We define the trace of a square $n \times n$ matrix $A = [a_{ij}]$ by

$$Tr(A) = \sum_{j=1}^{n} a_{jj}.$$

Another very important number associated with a square matrix A is the determinant of A.

Throughout this chapter, when we use the term matrix, we mean square matrix.

Definition 1: Let $S = \{1, 2, \ldots, n\}$ be the set of integers from 1 to n, arranged in ascending order. A rearrangement $j_1 j_2 \cdots j_n$ of the elements of S is called a permutation of S. We can consider a permutation of S to be a one-to-one mapping of S onto itself.

* To illustrate the preceding definition, let $S = \{1, 2, 3, 4\}$. Then 4231 is a permutation of S. It corresponds to the function $f: S \longrightarrow S$ defined by

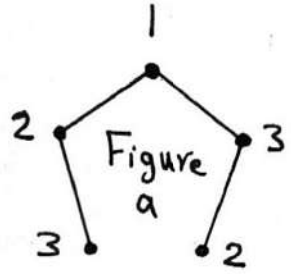
$$f(1) = 4$$
$$f(2) = 2$$
$$f(3) = 3$$
$$f(4) = 1.$$

We can put any one of the n elements of S in first position, any one of the remaining $n-1$ elements in second position, any one of remaining $n-2$ elements in third position, and so on until the nth position can be filled only by the last remaining element. Thus there are

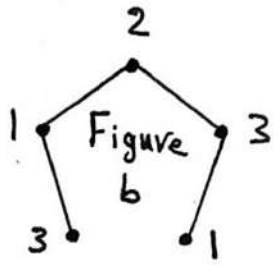
$n(n-1)(n-2)\cdots 2.1 = n!$ (n factorial) permutations of S; we denote the set of all permutations of S by S_n.

Example 2:

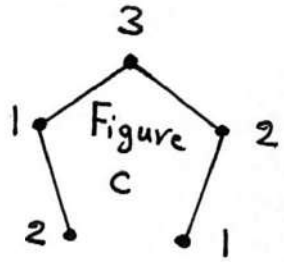
Let $S = \{1, 2, 3\}$. The set S_3 of all permutations of S consists of $3! = 3 \times 2 \times 1 = 6$ permutations 123, 132, 213, 231, 312, and 321.



123
132



213
231



312
321

} Permutations

Remarks 3: ① A permutation $j_1 j_2 \cdots j_n$ is said to have an inversion if a larger integer, j_r, precedes a smaller one, j_s.

② A permutation is called even if the total number of inversions in it is even, or it is called odd if the total number of inversions in it is odd.

Example 4: S_1 has only $1! = 1$ permutation: 1, which is even because there are no inversions.

Example 5: S_2 has $2! = 2$ permutations: 12, which is even (no inversions), and 21, which is odd (one inversion).

Example 6: In the permutation 4312 in S_4, 4 precedes 3, 4 precedes 1, 4 precedes 2, 3 precedes 1, and 3 precedes 2. Thus the total number of inversions in this permutation is 5, and 4312 is odd.

Example 7: S_3 has $3! = 3.2.1 = 6$ permutations: 123, 231, and 312, which are even, and 132, 213, and 321, which are odd.

Definition 8: Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant function, denoted by det, is defined by

$$\det(A) = \sum (\pm)\, a_{1j_1}\, a_{2j_2} \cdots a_{nj_n}, \quad \text{--- (1)}$$

where the summation is over all permutations $j_1 j_2 \cdots j_n$ of the set $S = \{1, 2, \cdots, n\}$. The sign is taken as + or − according to whether the permutation $j_1 j_2 \cdots j_n$ is even or odd.

Another notation for $\det(A)$ is $|A|$. We shall use both $\det(A)$ and $|A|$.

Example 9 : If $A = [a_{11}]$ is a 1×1 matrix, then $\det(A) = a_{11}$.

Example 10 : If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then to obtain $\det(A)$, we have $n=2$, So S_2 has 12 and 21 permutations and apply equation (1) we obtain

$$\det(A) = a_{11}\, a_{22} - a_{12}\, a_{21}.$$

In other word,

$$\begin{array}{ccc} a_{11} & & a_{12} \\ & + & \\ a_{21} & - & a_{22} \end{array}$$

Example 11 : If $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$, then $|A| = (2)(5) - (-3)(4) = 22$.

Example 12 : If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then to compute $\det(A)$, we have $n=3$, So S_3 has 123, 132, 213, 231, 312, and 321 permutations and apply equation (1) we obtain

$$\det(A) = a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{22}\, a_{31} \quad \cdots (2)$$

Example 13: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

Evaluate $|A|$.

Solution: Substituting in (2), we find that

$$|A| = (1)(1)(2) + (2)(3)(3) + (3)(2)(1) - (1)(3)(1) - (2)(2)(2) - (3)(1)(3) = 6.$$

Key Terms

Trace Permutation Determinant (det) Inrersion

Exercises:

Q.1 Evaluate 1) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$ 2) $\begin{vmatrix} 3 & 1 \\ -3 & -1 \end{vmatrix}$.

Q.2 Let $A = [a_{ij}]$ be a 4×4 matrix. Describe the general expression for $\det(A)$.

Q.3 Evaluate 1) $\det\left(\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)$ 2) $\begin{vmatrix} 2 & 1 & 3 \\ -3 & 2 & 1 \\ -1 & 3 & 4 \end{vmatrix}$.

Q.4 Evaluate $\begin{vmatrix} t & 4 \\ 5 & t-8 \end{vmatrix}$.

3.2 Properties of Determinants:

In this section we examine properties of determinants that simplify their computation.

Remarks 1:

a) If A is a matrix, then $\det(A) = \det(A^T)$.

Example a.1: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$. Then

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}.$$

Substituting in (2) of Section 3.1, we find that

$$|A^T| = (1)(1)(2) + (2)(1)(3) + (3)(2)(3) - (1)(1)(3) - (2)(2)(2) - (3)(1)(3) = 6 = |A|.$$

b) If matrix B results from matrix A by interchanging two different rows of A, then $\det(B) = -\det(A)$.

Example b.1: $|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = - \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}$

c) Theorem c.1: If two rows of A are equal, then $\det(A) = 0$

Proof: Suppose that rows r and s of A are equal. Interchange rows r and s of A to obtain a matrix B. Then $\det(B) = -\det(A)$. On the other hand, $B = A$, so $\det(B) = \det(A)$. Thus $\det(A) = -\det(A)$, and so $\det(A) = 0$.

Example c.2 : We have $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 0$. (Verify by the use of (2))

d) Theorem d.1 : If a row of A consists entirely of zeros, then $\det(A) = 0$

Proof : Let the ith row of A consist entirely of zeros. Since each term in Definition 8 for the determinant of A contains a factor from the ith row, each term in $\det(A)$ is zero. Hence $\det(A) = 0$.

Example d.2 : We have $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$. (Verify by the use of (2))

e) Theorem e.1 : If B is obtained from A by multiplying a row of A by real number k, then $\det(B) = k \det(A)$.

Proof : Suppose that the rth row of $A = [a_{ij}]$ is multiplied by k to obtain $B = [b_{ij}]$. Then $b_{ij} = a_{ij}$ if $i \neq r$ and $b_{rj} = k a_{rj}$. Using Definition 8, we obtain $\det(B)$ as

$$\begin{aligned} \det(B) &= \sum (\pm)\, b_{1j_1} b_{2j_2} \cdots b_{rj_r} \cdots b_{nj_n} \\ &= \sum (\pm)\, a_{1j_1} a_{2j_2} \cdots (k a_{rj_r}) \cdots a_{nj_n} \\ &= k \left(\sum (\pm)\, a_{1j_1} a_{2j_2} \cdots a_{rj_r} \cdots a_{nj_n} \right) \\ &= k \det(A). \end{aligned}$$

Example e.2 : We have $\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix}$

f) If $B = [b_{ij}]$ is obtained from $A = [a_{ij}]$ by adding to each element of the rth row of A, k times the corresponding element of the sth row, $r \neq s$, of A, then $\det(B) = \det(A)$.

Example f.1: We have

$$4 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix},$$

obtained by adding twice the second row to the first row

g) Theorem g.1: If a matrix $A = [a_{ij}]$ is upper (lower) triangular, then $\det(A) = a_{11}\, a_{22} \cdots a_{nn}$; that is, the determinant of a triangular matrix is the product of the elements on the main diagonal.

Proof:

Let $A = [a_{ij}]$ be upper triangular (that is, $a_{ij} = 0$ for $i > j$). Then a term $a_{1j_1}\, a_{2j_2} \cdots a_{nj_n}$ in the expression for $\det(A)$ can be nonzero only $1 \leq j_1$, $2 \leq j_2$, $\cdots$, $n \leq j_n$. Now $j_1 j_2 \cdots j_n$ must be a permutation, or rearrangement, of $\{1, 2, \cdots, n\}$. Hence we must have $j_1 = 1$, $j_2 = 2$, $\cdots$, $j_n = n$. Thus the only term of $\det(A)$ that can be nonzero is the product of the elements on the main diagonal of A. Hence $\det(A) = a_{11}\, a_{22} \cdots a_{nn}$.

Example 9.2 : Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ be an upper triangular matrix

Since $n = 3$, we have $S_3 = 3 \times 2 = 6$ permutations

123, 231, 312, 132, 213, and 321. By using Definition 8, Equation 1 to obtain the det (A)

$$\det(A) = \sum (\pm)\, a_{1j_1}\, a_{2j_2} \cdots a_{nj_n}$$

$$= a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32}$$
$$- a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} - a_{13}\, a_{22}\, a_{31}$$
$$= (1)(3)(2) + (3)(5)(0) + (3)(0)(0)$$
$$- (1)(5)(0) - (3)(0)(2) - (3)(3)(0)$$
$$= (1)(3)(2) = 6$$

* We can now interpret Property b (Remarks 1), Theorem 2.1 (Remarks 1), and Property f (Remarks 1) in terms of the following notation as follows :

$$\det\left(A_{r_i \leftrightarrow r_j}\right) = -\det(A), \quad i \neq j$$

$$\det\left(A_{kr_i \Rightarrow r_i}\right) = k \det(A),$$

$$\det\left(A_{kr_i + r_j \rightarrow r_j}\right) = \det(A), \quad i \neq j.$$

It is convenient to rewrite these properties in terms of $\det(A)$:

$$\det(A) = -\det\left(A_{r_i \leftrightarrow r_j}\right), \quad i \neq j$$

$$\det(A) = \frac{1}{k}\det\left(A_{kr_i \to r_i}\right), \quad k \neq 0$$

$$\det(A) = \det\left(A_{kr_i + r_j \to r_j}\right), \quad i \neq j.$$

Example 2: Let $A = \begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{bmatrix}$. Compute $\det(A)$.

Solution: We have

$$\det(A) = 2\det\left(A_{\frac{1}{2}r_3 \to r_3}\right)$$

$$= 2\det\left(\begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{bmatrix}\right)$$

$$= (-1)(2)\det\left(\begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{r_1 \leftrightarrow r_3}\right)$$

$$= -2\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 5 \\ 4 & 3 & 2 \end{bmatrix}\right)$$

$$= -2\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 5 \\ 4 & 3 & 2 \end{bmatrix}_{\substack{-3r_1 + r_2 \to r_2 \\ -4r_1 + r_3 \to r_3}}\right)$$

Zero out below the (1,1) entry.

$$= -2 \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & -5 & -10 \end{bmatrix} \right)$$

$$= -2 \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & -5 & -10 \end{bmatrix}_{-\frac{5}{8} r_2 + r_3 \to r_3} \right)$$ Zero out below the (2, 2) entry.

$$= -2 \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & 0 & -\frac{30}{4} \end{bmatrix} \right).$$

Next we compute the determinant of the upper triangular matrix.

$\det(A) = -2\,(1)(-8)\left(-\frac{30}{4}\right) = -120$, by Theorem 3.1.

Remark 3: The method used to compute a determinant in Example 2 will be referred to as computation via reduction to triangular form.

Remark 4: We can now compute the determinant of the identity matrix I_n: $\det(I_n) = 1$. We can also compute the determinants of the elementary matrices discussed in Section 2.3, as follows.

Let E_1 be an elementary matrix of type I. By Remarks 1, Property (b) we have that $\det(E_1) = -\det(I_n) = -1$.

Now let E_2 be an elementary matrix of type II. By Remarks 1, Theorem 2.1 we have that $\det(E_2) = k \det(I_n) = k$.

Finally, let E_3 be an elementary matrix of type III. By Remarks 1., Property f we have that $\det(E_3) = \det(I_n) = 1$.

Lemma 5: If E is an elementary matrix, then

$$\det(EA) = \det(E)\det(A),$$

and

$$\det(AE) = \det(A)\det(E).$$

Theorem 6: If A is an $n \times n$ matrix, then A is nonsingular if and only if $\det(A) \neq 0$.

Proof: If A is nonsingular, then A is a product of elementary matrices (Theorem 8). Thus let $A = E_1 E_2 \cdots E_k$. Then

$$\det(A) = \det(E_1 E_2 \cdots E_k) = \det(E_1)\det(E_2)\cdots\det(E_k) \neq 0.$$

Conversely, if A is singular, then A is row equivalent to a matrix B that has a row of zeros (A is singular iff A is row equivalent to a matrix B that has a row of zeros). Then $A = E_1 E_2 \cdots E_r B$, where $E_1, E_2, \ldots, E_r$ are elementary matrices. It then follows by the observation following Lemma 5 that

$$\det(A) = \det(E_1 E_2 \cdots E_r B) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B) = 0,$$

since $\det(B) = 0$.

Corollary 7: If A is an $n \times n$ matrix, then $Ax = 0$ has a nontrivial solution if and only if $\det(A) = 0$.

Example 8: Let A be a 4×4 matrix with $\det(A) = -2$.

1) Describe the set of all solutions to the homogeneous system $Ax = 0$.

2) If A is transformed to reduced row echelon form B, what is B?

3) Given an expression for a solution to the linear system $Ax = b$, where

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

4) Can the linear system $Ax = b$ have more than one solution? Explain.

5) Does A^{-1} exist?

Solution:

1) Since $\det(A) \neq 0$, by Corollary 7, the homogeneous system has only the trivial solution.

2) Since $\det(A) \neq 0$, by in Section 2.3, A is a nonsingular matrix, and by , $B = I_n$.

3) A solution to the given system is given by $x = A^{-1}b$.

4) No. The solution given in part (3) is the only one.

5) Yes.

Remark 9: If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Example 10 : Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Then $|A| = -2$ and $|B| = 5$.

On the other hand,

$$AB = \begin{bmatrix} 4 & 3 \\ 10 & 5 \end{bmatrix}, \text{ and } |AB| = -10 = |A||B|.$$

Remark 11 : If A is nonsingular, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Key Terms :

Properties of the determinant

Elementary matrix

Reduction to triangular form

Exercises :

Q.1 Compute the following determinants via reduction to triangular form or by citing a particular theorem, corollary or properties

1) $\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$ 2) $\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$.

Q.2 Verify that $\det(AB) = \det(A)\det(B)$ for the following

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \\ 2 & 1 & 3 \end{bmatrix}.$$

Q.3 Evaluate

1) $\begin{vmatrix} t-1 & -1 & -2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{vmatrix}$ 2) $\begin{vmatrix} t+1 & 4 \\ 2 & t-3 \end{vmatrix}$.

Q.4 If A is a nonsingular matrix such that $A^2 = A$, what is $\det(A)$?

Q.5 If $\det(A) = 2$, find $\det(A^5)$.

3.3 Cofactor Expansion:

We now develop a method for evaluating the determinant of $n \times n$ matrix that reduces the problem to the evaluation of determinants of matrices of order $n-1$. We can then repeat the process for these $(n-1) \times (n-1)$ matrices until we get to 2×2 matrices.

Definition 1: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and j^{th} column of A. The determinant $\det(M_{ij})$ is called the minor of a_{ij}.

Definition 2: Let $A = [a_{ij}]$ be an $n \times n$ matrix. The cofactor A_{ij} of a_{ij} is defined as $A_{ij} = (-1)^{i+j} \det(M_{ij})$.

Example 3: Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}.$$

Then

$$\det(M_{12}) = \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = 8 - 42 = -34,$$

and

$$\det(M_{23}) = \begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix} = 3 + 7 = 10,$$

also,

$$\det(M_{31}) = \begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix} = -6 - 10 = -16.$$

Also,

$$A_{12} = (-1)^{1+2} \det(M_{12}) = (-1)(-34) = 34,$$

$$A_{23} = (-1)^{2+3} \det(M_{23}) = (-1) \cdot 10 = -10$$

and

$$A_{31} = (-1)^{3+1} \det(M_{31}) = (1)(-16) = -16.$$

Remark 4: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in} \quad \cdots (1)$$

{ expansion of $\det(A)$ along the ith row }

Example 5: To evaluate the determinant

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}.$$

Solution:

We apply equation (1) with $i = 3$, we have

$$\det(A) = a_{31} A_{31} + a_{32} A_{32} + a_{33} A_{33} + a_{34} A_{34}$$

$$= a_{31} A_{31} + 0 \cdot A_{32} + 0 \cdot A_{33} + a_{34} A_{34}$$

$$= 3 \cdot A_{31} + (-3) \cdot A_{34}$$

$$= (3) \cdot (-1)^{3+1} \begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} + (-3) \cdot (-1)^{3+4} \begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

$$= (3)(1)(20) + (-3)(-1)(-4) = 48.$$

Key Terms:

Minor

Cofactor

Expansion along a row

Exercises:

Q.1 Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{bmatrix}$. Find the following minors:

1) $\det(M_{13})$ 2) $\det(M_{22})$ 3) $\det(M_{31})$ 4) $\det(M_{32})$.

Q.2 Let $A = \begin{bmatrix} -1 & 2 & 3 \\ -2 & 5 & 4 \\ 0 & 1 & -3 \end{bmatrix}$. Find the following cofactors:

1) A_{13} 2) A_{21} 3) A_{32} 4) A_{33}.

Q.3 Find all values of t for which

$$\begin{vmatrix} t-2 & 2 \\ 3 & t-3 \end{vmatrix} = 0$$

Q.4 Find all values of t for which

$$\begin{vmatrix} t-1 & 0 & 1 \\ -2 & t+2 & -1 \\ 0 & 0 & t+1 \end{vmatrix} = 0.$$

3.4 Inverse of a Matrix:

Remark 1: If $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$a_{i1} A_{k1} + a_{i2} A_{k2} + \cdots + a_{in} A_{kn} = 0 \quad \text{for } i \neq k;$$

Example 2: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & 5 & -2 \end{bmatrix}$. Then

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = 19,$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = -14$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 3.$$

Now for $i = 3$ and $k = 2$ we obtain

$$a_{31} A_{21} + a_{32} A_{22} + a_{33} A_{23} = (4)(19) + (5)(-14) + (-2)(3)$$
$$= 0,$$

for $i = 1$ and $k = 2$ we have

and $a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = (1)(19) + (2)(-14) + (3)(3)$
$$= 0.$$

* We may summarize our expansion results by writing

$$a_{i1} A_{k1} + a_{i2} A_{k2} + \cdots + a_{in} A_{kn} = \det(A) \text{ if } i = k$$
$$= 0 \quad \text{if } i \neq k.$$

Definition 3: Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $n \times n$ matrix $\operatorname{adj} A$, called the adjoint of A, is the matrix whose $(i,j)^{th}$ entry is the cofactor A_{ji} of a_{ji}. Thus

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}.$$

Example 4: Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$. Compute $\operatorname{adj} A$.

Solution: We first compute the cofactors of A. We have

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} = -18; \quad A_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} = 17;$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} = -6; \quad A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6;$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -10; \quad A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -2;$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} = -10; \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = -1;$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ 5 & 6 \end{vmatrix} = 28.$$ Then $\operatorname{adj} A = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$.

Theorem 5: If $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$A\,(\text{adj } A) = (\text{adj } A)\,A = \det(A)\, I_n .$$

Proof: We have

$$A\,(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{j1} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{j2} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{jn} & \cdots & A_{nn} \end{bmatrix} .$$

The $(i,j)^{th}$ element in the product matrix $A\,(\text{adj } A)$ is, by Remark 4,

$$a_{i1} A_{j1} + a_{i2} A_{j2} + \cdots + a_{in} A_{jn} = \begin{cases} \det(A), & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

This means that

$$A\,(\text{adj } A) = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \det(A) \end{bmatrix} = \det(A)\, I_n .$$

The $(i,j)^{th}$ element in the product matrix $(\text{adj } A)\,A$ is, by Remark 4,

$$A_{1i} a_{1j} + A_{2i} a_{2j} + \cdots + A_{ni} a_{nj} = \begin{cases} \det(A), & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Thus $(\text{adj } A)\,A = \det(A)\, I_n$.

Example 6: Consider the matrix of Example 4. Then

$$\begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix} = \begin{bmatrix} -94 & 0 & 0 \\ 0 & -94 & 0 \\ 0 & 0 & -94 \end{bmatrix}$$

$$= -94 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

And

$$\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix} = -94 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We now have a new method for finding the inverse of a nonsingular matrix, and we state this result as the following Corollary:

Corollary 7: If A is an $n \times n$ matrix and $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} (\text{adj } A) = \begin{bmatrix} \frac{A_{11}}{\det(A)} & \frac{A_{21}}{\det(A)} & \cdots & \frac{A_{n1}}{\det(A)} \\ \frac{A_{12}}{\det(A)} & \frac{A_{22}}{\det(A)} & \cdots & \frac{A_{n2}}{\det(A)} \\ \vdots & \vdots & & \vdots \\ \frac{A_{1n}}{\det(A)} & \frac{A_{2n}}{\det(A)} & \cdots & \frac{A_{nn}}{\det(A)} \end{bmatrix}.$$

Proof: By Theorem 5, $A\,(\text{adj } A) = \det(A)\, I_n$, so if $\det(A) \neq 0$, then

$$A\left(\frac{1}{\det(A)}(\text{adj } A)\right) = \frac{1}{\det(A)}\left(A(\text{adj}(A))\right)$$

$$= \frac{1}{\det(A)}\left(\det(A)\, I_n\right)$$

$$= I_n .$$

Hence

$$A^{-1} = \frac{1}{\det(A)}\ \text{adj}\,(A).$$

Example 7: Again consider the matrix of Example 4. Then $\det(A) = -94$, and

$$A^{-1} = \frac{1}{\det(A)}\ \text{adj}\,(A) = \begin{bmatrix} \frac{18}{94} & \frac{6}{94} & \frac{10}{94} \\ -\frac{17}{94} & \frac{10}{94} & \frac{1}{94} \\ \frac{6}{94} & \frac{2}{94} & -\frac{28}{94} \end{bmatrix}$$

Key Terms:

Inverse

Adjoint

Exercises:

Q.1 Let $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$.

1) Find $adj\ A$.

2) Compute $\det(A)$.

3) Show that $A\,(adj\ A) = (adj\ A)A = \det(A)\, I_3$.

Q.2 Find the inverse of the matrix in Q.1 by the method given in Corollary 7.

Q.3 Use the method given in Corollary 7 to find the inverse if it exists, of

$$A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ -2 & 1 & 5 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

3.5 Application of Determinants:

We can obtain another method for solving a linear system of n equations in n unknowns. This method is known as Cramer's rule.

Theorem 1: Cramer's Rule

Let

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

be a linear system of n equations in n unknowns, and let $A = [a_{ij}]$ be the coefficient matrix so that we can write the given system as $Ax = b$, where

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If $\det(A) \neq 0$, then the system has the unique solution

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \cdots, \quad x_n = \frac{\det(A_n)}{\det(A)},$$

where A_i is the matrix obtained from A by replacing the ith column of A by b.

Proof:

If $\det(A) \neq 0$, then by Theorem 6, A is nonsingular. Hence

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}b = \begin{bmatrix} \frac{A_{11}}{\det(A)} & \frac{A_{21}}{\det(A)} & \cdots & \frac{A_{n1}}{\det(A)} \\ \frac{A_{12}}{\det(A)} & \frac{A_{22}}{\det(A)} & \cdots & \frac{A_{n2}}{\det(A)} \\ \vdots & & & \\ \frac{A_{1i}}{\det(A)} & \frac{A_{2i}}{\det(A)} & \cdots & \frac{A_{ni}}{\det(A)} \\ \vdots & \vdots & & \vdots \\ \frac{A_{1n}}{\det(A)} & \frac{A_{2n}}{\det(A)} & \cdots & \frac{A_{nn}}{\det(A)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

This means that

$$x_i = \frac{A_{1i}}{\det(A)} b_1 + \frac{A_{2i}}{\det(A)} b_2 + \cdots + \frac{A_{ni}}{\det(A)} b_n \quad \text{for } i = 1, 2, \ldots, n.$$

Now let

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i-1} & b_1 & a_{1i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i-1} & b_2 & a_{2i+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni-1} & b_n & a_{ni+1} & \cdots & a_{nn} \end{bmatrix}.$$

If we evaluate $\det(A_i)$ by expanding along the cofactors of the ith column, we find that

$$\det(A_i) = A_{1i} b_1 + A_{2i} b_2 + \cdots + A_{ni} b_n.$$

Hence

$$x_i = \frac{\det(A_i)}{\det(A)} \quad \text{for } i = 1, 2, \ldots, n.$$

Example 2: Consider the following linear system:

$$\begin{aligned} -2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 4 \\ -2x_1 - x_2 + x_3 &= -3. \end{aligned}$$

Solution:

We have $|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2$. Then

$$x_1 = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{|A|} = \frac{-4}{-2} = 2,$$

$$x_2 = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{|A|} = \frac{-6}{-2} = 3,$$

and

$$x_3 = \frac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{|A|} = \frac{-8}{-2} = 4.$$

Concluding Remarks 3:

The following statements are equivalent for an $n \times n$ matrix A:

1) A is nonsingular.
2) $Ax = 0$ has only the trivial solution.
3) A is row equivalent to I_n.
4) The linear system $Ax = b$ has a unique solution for every $n \times 1$ matrix b.
5) A is a product of elementary matrices.
6) $\det(A) \neq 0$

Key Terms: Cramer's rule

Exercises:

Q.1 If possible, solve the following linear systems by Cramer's rule:

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 2 \\ x_1 + 2x_3 &= 0 \\ 2x_1 + 3x_2 - x_3 &= -5. \end{aligned}$$

Q.2 Solve the following linear system for x_3, by Cramer's rule:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 6 \\ 3x_1 + 2x_2 - 2x_3 &= -2 \\ x_1 + x_2 + 2x_3 &= -4. \end{aligned}$$

Q.3 Repeat Q.1 for the following linear system:

$$\begin{aligned} 2x_1 + 3x_2 + 7x_3 &= 0 \\ -2x_1 - 4x_3 &= 0 \\ x_1 + 2x_2 + 4x_3 &= 0. \end{aligned}$$

CHAPTER 4: Real Vector Spaces

Definition 1: A real vector space is a set V of elements on which we have two operations $+$ and $\cdot$ defined with the following properties:

a) If u and v are any elements in V, then $u+v$ is in V. (We say that V is closed under the operation $+$.)

1) $u+v = v+u$ for all u, v in V.

2) $u+(v+w) = (u+v)+w$ for all u, v, w in V.

3) There exists an element 0 in V such that $u+0 = 0+u = u$ for any u in V.

4) For each u in V there exists an element $-u$ in V such that $u + -u = -u + u = 0$.

b) If u is any element in V and c is any real number, then $c \cdot u$ is in V (i.e., V is closed under the operation $\cdot$).

5) $c \cdot (u+v) = c \cdot u + c \cdot v$ for any u, v in V and any real number c.

6) $(c+d) \cdot u = c \cdot u + d \cdot u$ for any u in V and any real numbers c and d.

7) $c \cdot (d \cdot u) = (cd) \cdot u$ for any u in V and any real numbers c and d.

8) $1 \cdot u = u$ for any u in V.

Remark 2: The elements of V are called vectors; the elements of the set of real numbers $\mathbb{R}$ are called scalars. The operation $+$ is called vector addition; the operation $\cdot$ is called scalar multiplication. The vector 0 in property (3) is called a zero vector. The vector $-u$ in property (4) is called a negative of u.

Example 2: Consider R^n, the set of all $n \times 1$ matrices

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

with real entries. Let the operation $+$ be matrix addition and let the operation $\cdot$ be multiplication of a matrix by a real number (scalar multiplicati

Solution:

a) Let $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be any vectors in R^n.

$$u + v = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} \in R^n$$

1) Let $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be any vectors in R^n

$$u + v = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} b_1 + a_1 \\ b_2 + a_2 \\ \vdots \\ b_n + a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = v + u.$$

2) Similarly to property (1), it is obvious to show that

$u + (v + w) = (u + v) + w$ for all u, v, w in R^n.

3) There exists an element $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ in R^n such that

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$ for any $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ in R^n.

4) Let $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ in R^n there exists an element $-u = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix}$ in R^n such that $u + (-u) = -u + u = 0$, because

$$u + (-u) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix} = \begin{bmatrix} a_1 - a_1 \\ a_2 - a_2 \\ \vdots \\ a_n - a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

Similarly for the case $-u + u = 0$.

b) Let $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ be any vector in R^n and c is any real number, then

$$c \cdot u = c \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ can \end{bmatrix} \in R^n$$

5) Let $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ in R^n and c is any real number, then

$$c \cdot (u+v) = c \cdot \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right) = c \cdot \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

$$= \begin{bmatrix} ca_1 + cb_1 \\ ca_2 + cb_2 \\ \vdots \\ can + cb_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ can \end{bmatrix} + \begin{bmatrix} cb_1 \\ cb_2 \\ \vdots \\ cb_n \end{bmatrix} = c \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + c \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= c \cdot u + c \cdot v.$$

6) Let $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ be any vector in R^n and c, d are any real numbers.

$$(c+d)\cdot\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} (c+d)a_1 \\ (c+d)a_2 \\ \vdots \\ (c+d)a_n \end{bmatrix} = \begin{bmatrix} ca_1 + da_1 \\ ca_2 + da_2 \\ \vdots \\ ca_n + da_n \end{bmatrix}$$

$$= \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} + \begin{bmatrix} da_1 \\ da_2 \\ \vdots \\ da_n \end{bmatrix} = c\cdot u + d\cdot u.$$

7) It is obvious to show that $c\cdot(d\cdot u) = (cd)\cdot u$ for any u in V and any real numbers c and d.

8) Let $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ be any vector in R^n

$$1\cdot u = 1\cdot\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Example 3: The set of all real numbers with + as the usual addition of real numbers and · as the usual multiplication of real numbers is a vector space (linear space). This vector space is essentially the case with $n=1$ of Example 2.

Example 4:

Let R_n be the set of all $1 \times n$ matrices $[a_1 \ a_2 \ \cdots \ a_n]$, where we define $+$ by

$$[a_1 \ a_2 \ \cdots \ a_n] \oplus [b_1 \ b_2 \ \cdots \ b_n] = [a_1+b_1 \ \ a_2+b_2 \ \cdots \ a_n+b_n]$$

and we define by

$$c \cdot [a_1 \ a_2 \ \cdots \ a_n] = [ca_1 \ \ ca_2 \ \cdots \ ca_n].$$ Then R_n is a vector space.

Example 5: Let V be the set of all 2×2 matrices with trace equal to zero; that is,

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in V provided $Tr(A) = a+d = 0$.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} r & s \\ t & p \end{bmatrix}$ be any elements of V. Then $Tr(A) = a+d = 0$ and $Tr(B) = r+p = 0$. For property (a), we have

$$A + B = \begin{bmatrix} a+r & b+s \\ c+t & d+p \end{bmatrix}$$

and

$$Tr(A+B) = (a+r) + (d+p) = (a+d) + (r+p) = 0+0 = 0,$$

So $A+B$ is in V; that is, V is closed under the operation $+$.

To verify property (3), observe that the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has trace equal to zero, so it is in V. Then it follows from the definition of $+$ that property (3) is valid in V.

To verify property (4), let A, as given previously, be an element of V and let

$$C = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}.$$

We first show that C is in V:

$$Tr(C) = -a + (-d) = -(a+d) = 0.$$

Then we have

$$A + C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0,$$

So $C = -A$. For property (b), let k be any real number. We have

$$k \cdot A = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

and

$$Tr(k \cdot A) = ka + kd = k(a+d) = 0,$$

So $k \cdot A$ is in V; that is, V is closed under the operation $\cdot$. For property (7) let k and m be any real numbers. Then

$$k \cdot (m \cdot A) = k \cdot \begin{bmatrix} ma & mb \\ mc & md \end{bmatrix} = \begin{bmatrix} kma & kmb \\ kmc & kmd \end{bmatrix}$$

and

$$(km) \cdot A = \begin{bmatrix} kma & kmb \\ kmc & kmd \end{bmatrix}.$$

It follows that $k \cdot (m \cdot A) = (km) \cdot A$.

Example 6: The set of all $m \times n$ matrices with matrix addition as $+$ and multiplication of a matrix by a real number as $\cdot$ is a vector space. We denote this vector space by M_{mn}.

Example 7 : Another source of examples are sets of polynomials ; therefore, we recall some well-known facts about such functions. A polynomial (in t) is a function that is expressible as

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 ,$$

where $a_0, a_1, \ldots, a_n$ are real numbers and n is a nonnegative integer.

Remarks 7.1

1) If $a_n \neq 0$, then $p(t)$ is said to have degree n. Thus the degree of a polynomial is the highest power of a term having a nonzero coefficient

2) $p(t) = 2t+1$ has degree 1, and the constant polynomial $p(t) = 3$ has degree 0.

3) The zero polynomial, denoted by $\mathbf{0}$, has no degree.

4) We now let P_n be the set of all polynomials of degree $\leq n$ together with the zero polynomial. If $p(t)$ and $q(t)$ are in P_n, we can write

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

and

$$q(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0 .$$

We define $p(t) + q(t)$ as

$$p(t) + q(t) = (a_n + b_n) t^n + (a_{n-1} + b_{n-1}) t^{n-1} + \cdots + (a_1 + b_1) t + (a_0 + b_0)$$

If c is a scalar, we also define $c \cdot p(t)$ as

$$c \cdot p(t) = (ca_n) t^n + (ca_{n-1}) t^{n-1} + \cdots + (ca_1) t + (ca_0) .$$

We now show that $\mathcal{P}_n$ is a vector space (linear space).

Let $p(t)$ and $q(t)$, as before, be elements of $\mathcal{P}_n$; that is, they are polynomials of degree $\leq n$ or the zero polynomial. Then the previous definitions of the operations $+$ and $\cdot$ show that $p(t) + q(t)$ and $c \cdot p(t)$, for any scalar c, are polynomials of degree $\leq n$ or the zero polynomial. That is, $p(t) + q(t)$ and $c \cdot p(t)$ are in $\mathcal{P}_n$ so properties (a) and (b) in Definition 1 hold. To verify property (1), we observe that

$$q(t) + p(t) = (b_n + a_n)t^n + (b_{n-1} + a_{n-1})t^{n-1} + \cdots + (b_1 + a_1)t + (a_0 + b_0),$$

and since $a_i + b_i = b_i + a_i$ holds for the real numbers, we conclude that $p(t) + q(t) = q(t) + p(t)$. Similarly, we verify property (2). The zero polynomial is the element 0 needed in property (3). If $p(t)$ is as given previously, then its negative, $-p(t)$, is

$$-a_n t^n - a_{n-1} t^{n-1} - \cdots - a_1 t - a_0.$$

We shall now verify property (6) and will leave the verification of the remaining properties to the reader. Thus

$$\begin{aligned}(c+d) \cdot p(t) &= (c+d)a_n t^n + (c+d)a_{n-1} t^{n-1} + \cdots + (c+d)a_1 t + (c+d)a_0 \\ &= c a_n t^n + d a_n t^n + c a_{n-1} t^{n-1} + d a_{n-1} t^{n-1} + \cdots + \\ &\quad c a_1 t + d a_1 t + c a_0 + d a_0 \\ &= c(a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0) \\ &\quad + d(a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0) \\ &= c \cdot p(t) + d \cdot p(t).\end{aligned}$$

Example 8: Let V be the set of all real numbers with the operations $u+v = u-v$ ($+$ is ordinary subtraction) and $c \cdot u = cu$ ($\cdot$ is ordinary multiplication). Is V a vector space? If it is not, which properties in Definition fail to hold?

Solution: If u and v are in V, and c is a scalar, then $u+v$ and $c \cdot u$ are in V, so that (a) and (b) in Definition hold. However, property (1) fails to hold, since

$$u + v = u - v \quad \text{and} \quad v + u = v - u,$$

and these are not the same, in general. (Find u and v such that $u - v \neq v - u$.)

The properties (2), (3), and (4) fail to hold. Also property (6) does not hold, because

$$(c+d) \cdot u = (c+d)u = cu + du,$$

whereas

$$c \cdot u + d \cdot u = cu + du = cu - du,$$

and these are not equal, in general. Thus V is not a vector space.

Example 9: Let V be the set of all integers; define $+$ as ordinary addition and $\cdot$ as ordinary multiplication. V is not a vector space, because if u is any nonzero vector in V and $c = \sqrt{3}$, then $c \cdot u = \sqrt{3}u$ is not in V. Thus (b) fails to hold.

Remark 10: To verify that a given set V with two operations $+$ and $\cdot$ is a real vector space, we must show that it satisfies all the properties of Definition 1. The first thing to check is whether (a) and (b) hold, for if either of these fails, we do not have a vector space. If both (a) and (b) hold, it is recommended that (3), the existence of zero element, be verified next. Naturally, if (3) fails to hold, we do not have a vector space and do not have to check the remaining properties.

The following theorem presents some useful properties common to all vector spaces:

Theorem 11: If V is a vector space, then

1) $0 \cdot u = \mathbf{0}$ for any vector u in V.

2) $c \cdot \mathbf{0} = \mathbf{0}$ for any scalar c.

3) If $c \cdot u = \mathbf{0}$, then either $c = 0$ or $u = \mathbf{0}$.

4) $(-1) \cdot u = -u$ for any vector u in V.

Proof: 1) We have

$$0 \cdot u = (0+0) \cdot u = 0 \cdot u + 0 \cdot u \quad \text{--- (1)}$$

by (6) of Definition . Adding $-0 \cdot u$ to both sides of Equation (1), we obtain by (2), (3), and (4) of Definition ,

$$0 \cdot u = \mathbf{0}.$$

3) Suppose that $c \cdot u = \mathbf{0}$. If $c = 0$ return to case property 1). So, we assume that $c \neq 0$. Multiply the equation by c^{-1}, we have

$c^{-1}(cv) = c^{-1}\mathbf{0}$, therefore, by associativity property of Definition we have $(c^{-1}c)u = \mathbf{0}$. Therefore $1u = \mathbf{0}$ and so $u = 0$.

4) $(-1) \cdot u + u = (-1) \cdot u + 1 \cdot u = (-1+1) \cdot u = 0 \cdot u = \mathbf{0}$.

Since $-u$ is unique, we conclude that

$$(-1) \cdot u = -u.$$

Remark 12: The following table summarizes that notation and the descriptions of the set:

R^n	the set of $n \times 1$ matrices.
R_n	the set of $1 \times n$ matrices.
M_{mn}	the set of $m \times n$ matrices.
P_n	the set of all polynomials of degree n or less together with the zero polynomial.

Key Terms:

Real vector space
Vector
Vector addition
Scalar multiplication
closed
Zero vector
Polynomial
Degree of a Polynomial
Zero polynomial

Exercises:

Q.1 The set of all ordered triples of real numbers with the operations

$$(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$$

and

$$r \cdot (x, y, z) = (x, 1, z).$$

Does this set form a vector space?

Q.2 Let V be the set of all positive real numbers; define $+$ by $u + v = uv - 1$ and $c \cdot v = v$. Is V a vector space?

Q.3 Prove that $-(-v) = v$.

Q.4 Let V be the set of all 2×2 matrices $A = \begin{bmatrix} a & b \\ 2b & d \end{bmatrix}$. Let the operation $+$ be addition of matrices and the operation $\cdot$ be the scalar multiplication of matrices.

1) Is V closed under addition.
2) Is V closed under scalar multiplication.
3) What is the zero vector in the set V.

4.3 Subspaces:

Definition 1: Let V be a vector space and W a nonempty subset of V. If W is a vector space with respect to the operations in V, then W is called a subspace of V.

Remark 2: It follows from Definition 1 that to verify that a subset W of a vector space V is a subspace, one must check that (a), (b), and (1) through (8) of Definition 1 hold. However, the next theorem says that it is enough to merely check that (a) and (b) hold to verify that a subset W of a vector space V is a subspace.

Theorem 3: Let V be a vector space with operations $+$ and $\cdot$ and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold:

1) If u and v are any vectors in W, then $u+v$ is in W.
2) If c is any real number and u is any vector in W, then $c \cdot u$ is in W.

Proof: If W is a subspace of V, then it is a vector space and 1) and 2) of Definition 1 hold; these are precisely 1) and 2) of the theorem.

Conversely, suppose that 1) and 2) hold. We wish to show that W is a subspace of V. First, from 2) we have that $(-1) \cdot u$ is in W for any u in W. From 1) we have that $u + (-1) \cdot u$ is in W. But $u + (-1) \cdot u = 0$, so 0 is in W. Then $u + 0 = u$ for any u in W. Finally, properties 1), 2), 5), 6), 7), and 8) hold in W because they hold in V. Hence W is a subspace of V

Example 4: Every vector space has at least two subspaces, itself and the subspace $\{\mathbf{0}\}$ consisting only of the zero vector. (Recall that $\mathbf{0}+\mathbf{0}=\mathbf{0}$ and $c\cdot\mathbf{0}=\mathbf{0}$ in any vector space.) Thus $\{0\}$ is a subspace of V. The subspace $\{\mathbf{0}\}$ is called the zero subspace of V.

Example 5: Let P_2 be the set consisting of all polynomials of degree ≤ 2 and the zero polynomial; P_2 is a subset of P, the vector space of all polynomials. To verify that P_2 is a subspace of P, show it is closed for $+$ and $\cdot$. In general, the set P_n consisting of all polynomials of degree $\leq n$ and the zero polynomial is a subspace of P. Also, P_n is a subspace of P_{n+1}.

Example 6: Let W be the set of all polynomials of degree exactly $= 2$; W is a subset of P, but W is not a subspace of P, because the sum of the polynomials $2t^2+3t+1$ and $-2t^2+t+2$ is not in W, since it is a polynomial of degree 1.

Example 7: Which of the following subsets of R^2 with the usual operations of the vector addition and scalar multiplication are subspaces?

1) W_1 is the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0$.

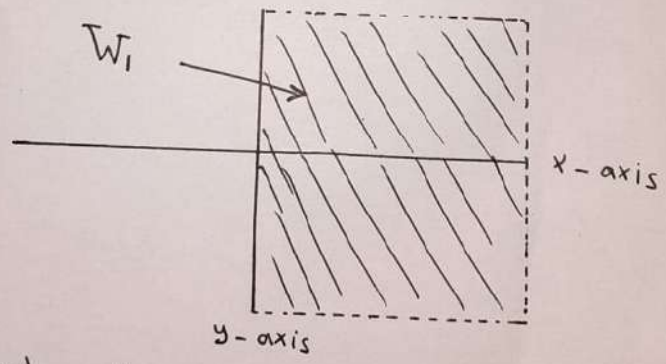
2) W_2 is the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0$, $y \geq 0$.

3) W_3 is the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x = 0$.

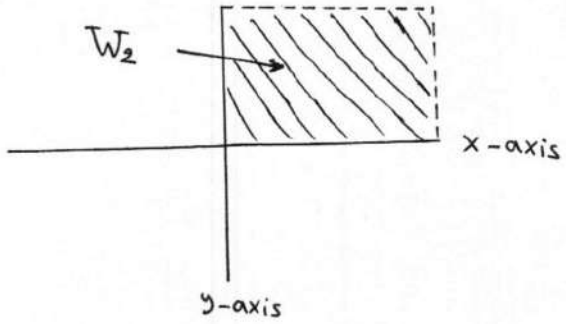
Solution:

1) W_1 is the right half of the xy-plane. It is not a subspace of $\mathbb{R}^2$, because if we take the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in W_1, then the scalar multiple

$$-3 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \end{bmatrix}$$ is not in W_1, so property 2) in Theorem 3 does not hold.



2) W_2 is the first quadrant of the xy-plane. The same vector and scalar multiple as in part 1) shows that W_2 is not a subspace.



3) W_3 is the y-axis in the xy-plane. To see whether W_3 is the subspace, let

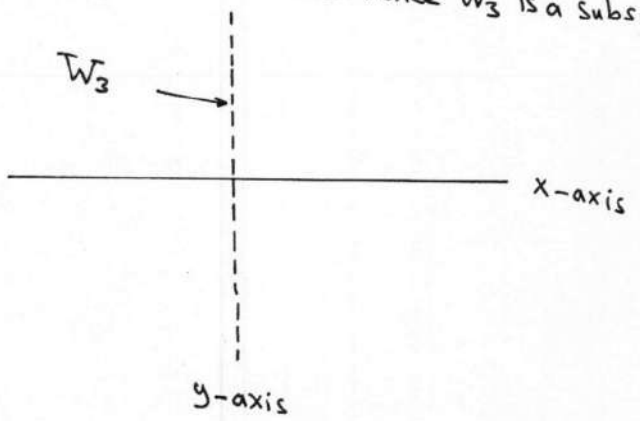
$$u = \begin{bmatrix} 0 \\ b_1 \end{bmatrix} \text{ and } v = \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \text{ be vectors in } W_3. \text{ Then}$$

$$u + v = \begin{bmatrix} 0 \\ b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 + b_2 \end{bmatrix}, \text{ which is in } W_3,$$

So property 1) in Theorem 3 holds. Moreover, if c is a scalar, then

$$c \cdot u = c \cdot \begin{bmatrix} 0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ cb_1 \end{bmatrix}, \text{ which is in } W_3,$$

So property 2) in Theorem 3 holds. Hence W_3 is a subspace of R^2.



Example 8: Let W be the set of all vectors in R^3 of the form

$\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$, where a and b are any real numbers. Prove that W is a subspace of R^3.

Solution: To verify Theorem 1) and 2), we let

$$u = \begin{bmatrix} a_1 \\ b_1 \\ a_1+b_1 \end{bmatrix} \text{ and } v = \begin{bmatrix} a_2 \\ b_2 \\ a_2+b_2 \end{bmatrix} \text{ be two vectors in } W. \text{ Then}$$

$$u+v = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ (a_1+b_1)+(a_2+b_2) \end{bmatrix} = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ (a_1+a_2)+(b_1+b_2) \end{bmatrix},$$

is in W, for W consists of all those vectors whose third entry is the sum of the first two entries. Similarly,

$$c \cdot \begin{bmatrix} a_1 \\ b_1 \\ a_1+b_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ cb_1 \\ c(a_1+b_1) \end{bmatrix} = \begin{bmatrix} ca_1 \\ cb_1 \\ ca_1+cb_1 \end{bmatrix}$$

is in W. Hence W is a subspace of R^3.

Remark 9: Henceforth, we shall show that a nonempty subset W of a vector space V is a subspace of V if and only if ---------- for any vectors u and v in W and any scalars c and d.

Definition 10: Let $v_1, v_2, \ldots, v_k$ be vectors in a vector space V. A vector v in V is called a linear combination of $v_1, v_2, \ldots, v_k$ if

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = \sum_{j=1}^{k} a_j v_j$$

for some real numbers $a_1, a_2, \ldots, a_k$.

Remark 11: Definition 10 was stated for a finite set of vectors, but it also applies to an infinite set S of vectors in a vector space using corresponding notation for infinite sums.

Example 12: In Example 8 we showed that W, the set of all vectors in R^3 of the form $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$, where a and b are any real numbers, is a subspace of R^3. Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Then every vector in W is a linear combination of v_1 and v_2, since

$$\begin{bmatrix} a \\ b \\ a+b \end{bmatrix} = a v_1 + b v_2.$$

Example 13: In Example 5, P_2 was the vector space of all polynomials of degree 2 or less and the zero polynomial. Every vector in P_2 has the form $at^2 + bt + c$, so each vector in P_2 is a linear combination of t^2, t, and 1.

Example 14: In R^3, let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The vector

$$v = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

is a linear combination of v_1, v_2, and v_3 if we can find real numbers a_1, a_2, and a_3 so that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = v.$$

Substituting for v, v_1, v_2, and v_3, we have

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}.$$

We obtain

$$\begin{aligned} a_1 + a_2 + a_3 &= 2 \\ 2a_1 \quad\quad + a_3 &= 1 \\ a_1 + 2a_2 \quad\quad &= 5. \end{aligned}$$

Solving this linear system by the methods of Chapter 2 gives $a_1 = 1$, $a_2 = 2$, and $a_3 = -1$, which means that v is a linear combination of v_1, v_2, and v_3.

$$v = v_1 + 2v_2 - v_3.$$

Example 15: Recall that if A is an $m \times n$ matrix, then the homogeneous system of m equations in n unknowns with coefficient matrix A can be written as

$$Ax = 0,$$

where x is a vector in R^n and 0 is the zero vector. Thus the set W of all solutions is a subset of R^n. We now show that W is a subspace of R^n (called the solution space of the homogeneous system, or the null space of the matrix A) by verifying 1) and 2) of Theorem 3. Let x and y be any solutions. Then

$$Ax = 0 \quad \text{and} \quad Ay = 0.$$

Now

$$A(x+y) = Ax + Ay = 0 + 0 = 0,$$

So $x+y$ is a solution. Also, if c is a scalar, then

$$A(cx) = c(Ax) = c0 = 0,$$

So cx is a solution. Thus W is closed under addition and scalar multiplication of vectors and is therefore a subspace of R^n.

Remark 16: It should be noted that the set of all solutions to the linear system $Ax = b$, $b \neq 0$, is not a subspace of R^n.

Key Terms:

Subspaces Subset Null (Solution) Space

Zero Subspace Linear combination

Exercises:

Q.1 The set W consisting of all the points in R^2 of the form (x, x) is a straight line. Is W a subspace of R^2? Explain.

Q.2 Which of the given subsets of R^3 are subspaces?
The set of all vectors of the form

1) $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$ 2) $\begin{bmatrix} a \\ b \\ a+2b \end{bmatrix}$ 3) $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ 4) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a + 2b - c = 0$

Q.3 Is a subset $[a\ b\ c\ d]$ is a subspace of R_4, where $a - b = 2$.

Q.4 Let W be the set of all 3x3 matrices of the form

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}.$$

Show that W is a subspace of M_{33}.

Q.5 Prove that P_2 is a subspace of P_3.

Q.6 If A is a nonsingular matrix, what is the null space of A?

Q.7 Which of the following vectors in R^3 are linear combinations of $v_1 = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, and $v_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$?

1) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ 2) $\begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}$ 3) $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ 4) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

4.4 Span :

Definition 1: If $S = \{v_1, v_2, \ldots, v_k\}$ is a set of vectors in a vector space V, then the set of all vectors in V that are linear combinations of the vectors in S is denoted by

Span S or span $\{v_1, v_2, \ldots, v_k\}$.

Remark 2: Definition 1 is stated for a finit set of vectors, but it also applies to an infinite set S of vectors in a vector space.

Example 3: Consider the set S of 2×3 matrices given by

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Let $v = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ be any vector in M_{23}.

$$a.\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b.\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \\ x' & y' & z' \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & d \end{bmatrix} = \begin{bmatrix} x & y & z \\ x' & y' & z' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix} = \begin{bmatrix} x & y & z \\ x' & y' & z' \end{bmatrix}.$$ We have

$x = a$, $y = b$, $z = 0$, $x' = 0$, $y' = c$, $z' = d$. That is, Span S is the subset of M_{23} consisting of all matrices of the form

$$\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}$$, where a, b, c, and d are real numbers.

Example 4: Let $S = \{t^2, t, 1\}$ be a subset of P_2. We have Span $S = P_2$. Since for arbitrary element $xt^2 + yt + z$ in P_2, $at^2 + bt + c = xt^2 + yt + z$, we obtain $x = a$, $y = b$, $z = c$, hence Span S is all polynomials of the form

$$at^2 + bt + c,$$

where a, b, and c are real numbers.

Example 5: Let $S = \left\{\begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ be a subset of R^3.

Let $v = \begin{bmatrix}x\\y\\z\end{bmatrix}$ be any vector in R^3. We have

$$a \cdot \begin{bmatrix}2\\0\\0\end{bmatrix} + b \cdot \begin{bmatrix}0\\-1\\0\end{bmatrix} + c \cdot \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2a\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-b\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2a\\-b\\0\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}. \text{ Hence } x = 2a,\ y = -b, \text{ and } z = 0.$$

Thus Span S is the set of all vectors in R^3 of the form

$$\begin{bmatrix}2a\\-b\\0\end{bmatrix},$$

where a and b are any real numbers.

Theorem 6: Let $S = \{ v_1, v_2, \dots, v_k \}$ be a set of vectors in a vector space V. Then span S is a subspace of V.

Proof: Let

$$u = \sum_{j=1}^{k} a_j v_j \quad \text{and} \quad v = \sum_{j=1}^{k} b_j v_j$$

for some real numbers $a_1, a_2, \dots, a_k$ and $b_1, b_2, \dots, b_k$. We have

$$u + v = \sum_{j=1}^{k} a_j v_j + \sum_{j=1}^{k} b_j v_j = \sum_{j=1}^{k} (a_j + b_j) v_j,$$

moreover, for any real number c,

$$cu = c\left(\sum_{j=1}^{k} a_j v_j \right) = \sum_{j=1}^{k} (c a_j) v_j.$$

Since the sum $u + w$ and the scalar multiple cu are linear combinations of the vectors in S, then span S is a subspace of V.

Example 7: Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be a subset of the vector space M_{22}. Then span S is the subspace of all 2×2 diagonal matrices.

* In order to completely describe a vector space V, we use the concept of span as stated in the following definition:

Definition 8: Let S be a set of vectors in a vector space V. If every vector in V is a linear combination of the vectors in S, then the set S is said to span V, or V is spanned by the set S; that is, span $S = V$.

Remark 9: If span $S = V$, S is called a spanning set of V. A vector space can have many spanning sets. In our examples we used sets S containing a finite number of vectors, but some vector spaces need spanning sets with infinitely many vectors. See Example 10.

Example 10: Let P be the vector space of all polynomials. Let $S = \{1, t, t^2, \dots\}$; that is, the set of all (nonnegative integer) powers of t. Then span $S = P$. Every spanning set for P will have infinitely many vectors.

* Another type of question that we encounter is; for a given subset S of a vector space V, is the vector w of V in span S? We will show in the following examples that for sets S with a finite number of vectors we can answer such questions by solving a linear system of equations:

Example 11: In R^3, let

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Determine whether the vector

$$v = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$$

belongs to span $\{v_1, v_2\}$

Solution: If we can find scalars a_1 and a_2 such that

$$a_1 v_1 + a_2 v_2 = v,$$

then v belongs to span $\{v_1, v_2\}$.

Substituting for v_1, v_2, and v, we have

$$a_1\begin{bmatrix}2\\1\\1\end{bmatrix} + a_2\begin{bmatrix}1\\-1\\3\end{bmatrix} = \begin{bmatrix}1\\5\\-7\end{bmatrix}.$$ We have

$$\begin{aligned} 2a_1 + a_2 &= 1 \\ a_1 - a_2 &= 5 \quad \cdots (1) \\ a_1 + 3a_2 &= -7. \end{aligned}$$

The augmented matrix of Equation (1) is

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -1 & 5 \\ 1 & 3 & -7 \end{array}\right].$$

The reduced row echelon form of this system is

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right],$$

which indicates that the linear system is consistent, $a_1 = 2$, $a_2 = -3$. Hence v belongs to span $\{v_1, v_2\}$.

Example 12: In P_2, let

$v_1 = 2t^2 + t + 2$, $v_2 = t^2 - 2t$, $v_3 = 5t^2 - 5t + 2$,

$v_4 = -t^2 - 3t - 2$. Determine whether the vector

$$v = t^2 + t + 2$$

belongs to span $\{v_1, v_2, v_3, v_4\}$.

Solution: If we can find scalars a_1, a_2, a_3, and a_4 so that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = v,$$

then v belongs to span $\{v_1, v_2, v_3, v_4\}$. Substituting for v_1, v_2, v_3, v_4, and v, we have

$$a_1(2t^2 + t + 2) + a_2(t^2 - 2t) + a_3(5t^2 - 5t + 2) + a_4(-t^2 - 3t - 2) = t^2 + t + 2,$$

or

$$(2a_1 + a_2 + 5a_3 - a_4)t^2 + (a_1 - 2a_2 - 5a_3 - 3a_4)t + (2a_1 + 2a_3 - 2a_4) = t^2 + t + 2.$$

Now two polynomials agree for all values of t only if the coefficients of respective powers of t agree. Thus we get the linear system

$$\begin{aligned} 2a_1 + a_2 + 5a_3 - a_4 &= 1 \\ a_1 - 2a_2 - 5a_3 - 3a_4 &= 1 \\ 2a_1 \qquad + 2a_3 - 2a_4 &= 2. \end{aligned}$$

To determine whether this system of linear equations is consistent, we form the augmented matrix and transform it to reduced row echelon form, obtaining

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

which indicates that the system is inconsistent; that is, it has no solution. Hence v does not belong to span $\{v_1, v_2, v_3, v_4\}$.

Example 13: Let V be the vector space R^3. Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \text{ and } \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

To find out whether v_1, v_2, v_3 span V, we pick any vector $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in V (where a, b, and c are arbitrary real numbers) and determine whether there are constants a_1, a_2, and a_3 such that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = v.$$

This leads to the linear system

$$\begin{aligned} a_1 + a_2 + a_3 &= a \\ 2a_1 \quad\quad + a_3 &= b \\ a_1 + 2a_2 \quad\quad &= c. \end{aligned}$$

A solution is

$$a_1 = \frac{-2a + 2b + c}{3}, \quad a_2 = \frac{a - b + c}{3}, \quad a_3 = \frac{4a - b - 2c}{3}$$

Thus v_1, v_2, v_3 span V. This is equivalent to saying that

$$\text{Span}\{v_1, v_2, v_3\} = R^3.$$

Key Terms:

Span of a set Set S spans vector space V V is spanned by the set S

Consistent system Inconsistent system

Exercises:

Q.1 For each of the following vector spaces, give spanning set:

1) R^3 2) M_{22} 3) P_2

Q.2 In each part, explain why the set S is not a spanning set for the vector space V.

1) $S = \{t^3, t^2, t\}$, $V = P_3$.

2) $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, $V = R^2$.

Q.3 In each part, determine whether the given vector $p(t)$ in P_2 belongs to span $\{P_1(t), P_2(t), P_3(t)\}$, where

$P_1(t) = t^2 + 2t + 1$, $P_2(t) = t^2 + 3$, and $P_3(t) = t - 1$.

1) $p(t) = t^2 + t + 2$ 2) $p(t) = -t^2 + t - 4$

Q.4 Which of the following sets of polynomials span P_2

1) $\{t^2+1, t^2+t, t+1\}$ 2) $\{t^2+1, t-1, t^2+t\}$ 3) $\{t^2+2t-1, t^2-1\}$

4) $\{t^2+2, 2t^2-t+1, t+2, t^2+t+4\}$.

Mathcad

User's Guide

Mathcad 2000 Professional
Mathcad 2000 Standard

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MathSoft
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How to Use This *User's Guide*

This *User's Guide* is organized into the following parts:

- The Basics

  This section contains a quick introduction to Mathcad's features and workspace, including resources available in the product and on the Internet for getting more out of Mathcad. Be sure to read this section first if you are a new Mathcad user.

- Creating Mathcad Worksheets

  This section describes in more detail how to create and edit Mathcad worksheets. It leads you through editing and formatting equations, text, and graphics, as well as opening, editing, saving, and printing Mathcad worksheets and templates.

- Computational Features

  This section describes how Mathcad interprets equations and explains Mathcad's computational features: units of measurement, complex numbers, matrices, built-in functions, solving equations, programming, and so on. This section also describes how to do symbolic calculations and how to use Mathcad's two- and three-dimensional plotting features.

The *User's Guide* ends with reference appendices and a comprehensive index.

As far as possible, the topics in this guide are described independently of each other. This means that once you are familiar with the basic workings of Mathcad, you can just select a topic of interest and read about it.

The on-line Mathcad Resource Center (choose **Resource Center** from the **Help** menu) provides step by step tutorials, examples, and application files that you can use directly in your own Mathcad worksheets. Mathcad QuickSheets are templates available in the Resource Center that provide live examples that you can manipulate.

Notations and Conventions

This *User's Guide* uses the following notations and conventions:

Italics represent scalar variable names, function names, and error messages.

`Bold Courier` represents keys you should type.

Bold represents a menu command. It is also used to denote vector and matrix valued variables.

An arrow such as that in "**Graph⇒X-Y Plot**" indicates a pull-right menu command.

Function keys and other special keys are enclosed in brackets. For example, [↑], [↓], [←], and [→] are the arrow keys on the keyboard. [**`F1`**], [**`F2`**], etc., are function keys; [**`BkSp`**] is the Backspace key for backspacing over characters; [**`Del`**] is the Delete key for deleting characters to the right; [**`Ins`**] is the Insert key for inserting characters to the left of the insertion point; [**`Tab`**] is the Tab key; and [**`Space`**] is the space bar.

[**`Ctrl`**], [**`Shift`**], and [**`Alt`**] are the Control, Shift, and Alt keys. When two keys are shown together, for example, [**`Ctrl`**]**`V`**, press and hold down the first key, and then press the second key.

The symbol [↵] and [**Enter**] refer to the same key.

When this *User's Guide* shows spaces in an equation, you need not type the spaces. Mathcad automatically spaces the equation correctly.

Pro This *User's Guide* applies to Mathcad 2000 Professional and Mathcad 2000 Standard. If you're not using Mathcad 2000 Professional, certain features described in the *User's Guide* will not be available to you. The word Pro appears:

- In the page margin, as it does above, whenever a section in a chapter describes a feature or a function that is unique to Mathcad 2000 Professional.
- In the page footer, whenever all features described in that chapter are unique to Mathcad 2000 Professional.

This *User's Guide* also describes a few product features that are available only in add-on packages for Mathcad. For example, some numerical solving features and functions are provided only in the *Solving and Optimization Extension Pack* (*Expert Solver).*

Chapter 2
Getting Started with Mathcad

- The Mathcad Workspace
- Regions
- A Simple Calculation
- Definitions and Variables
- Entering Text
- Iterative Calculations
- Graphs
- Saving, Printing, and Exiting

The Mathcad Workspace

For information on system requirements and how to install Mathcad on your computer, refer to Chapter 1, “Welcome to Mathcad.”

When you start Mathcad, you’ll see a window like that shown in Figure 2-1. By default the worksheet area is white. To select a different color, choose **Color⇒Background** from the **Format** menu.

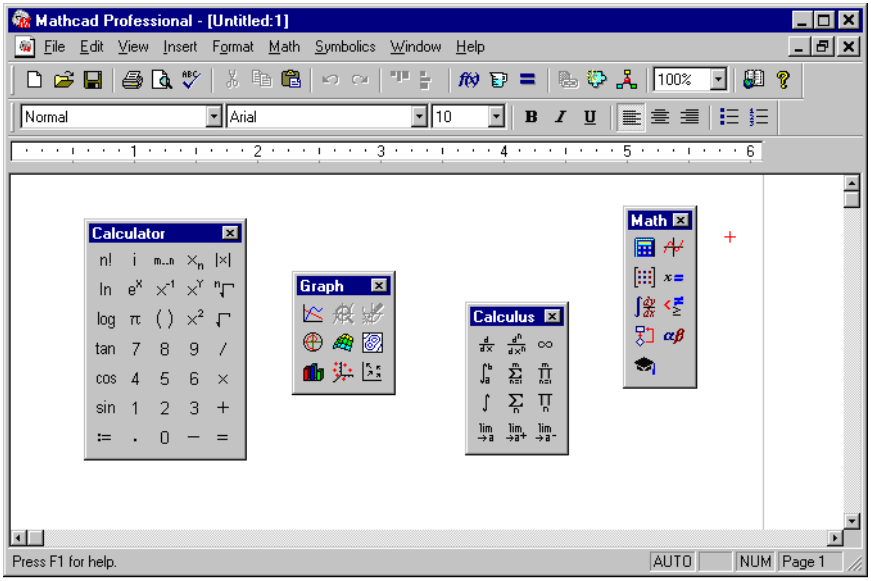
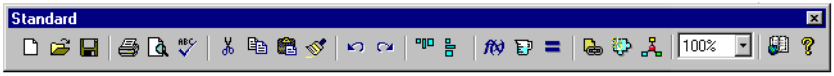


Figure 2-1: Mathcad Professional with various toolbars displayed.

Each button in the **Math toolbar**, shown in Figure 2-1, opens another toolbar of operators or symbols. You can insert many operators, Greek letters, and plots by clicking the buttons found on these toolbars:

Button	Opens math toolbar...
	**Calculator**—Common arithmetic operators.
	**Graph**—Various two- and three-dimensional plot types and graph tools.
	**Matrix**—Matrix and vector operators.
	**Evaluation**—Equal signs for evaluation and definition.
	**Calculus**—Derivatives, integrals, limits, and iterated sums and products.
	**Boolean**—Comparative and logical operators for Boolean expression.
	**Programming**—Programming constructs (*Mathcad Professional only*).
	**Greek**—Greek letters.
	**Symbolic**—Symbolic keywords.

The **Standard toolbar** is the strip of buttons shown just below the main menus in Figure 2-1:



Many menu commands can be accessed more quickly by clicking a button on the Standard toolbar.

The **Formatting toolbar** is shown immediately below the Standard toolbar in Figure 2-1. This contains scrolling lists and buttons used to specify font characteristics in equations and text.



Tip To learn what a button on any toolbar does, let the mouse pointer rest on the button momentarily. You'll see a tooltip beside the pointer giving a brief description.

To conserve screen space, you can show or hide each toolbar individually by choosing the appropriate command from the **View** menu. You can also detach and drag a toolbar around your window. To do so, place the mouse pointer anywhere other than on a button or a text box. Then press and hold down the mouse button and drag. You'll find that the toolbars rearrange themselves appropriately depending on where you drag them. And Mathcad remembers where you left your toolbars the next time you open the application.

Tip The Standard, Formatting, and Math toolbars are customizable. To add and remove buttons from one of these toolbars, click with the right mouse button on the toolbar and choose **Customize** from the pop-up menu to bring up the Customize Toolbar dialog box.

The **worksheet ruler** is shown towards the top of the screen in Figure 2-1. To hide or show the ruler, choose **Ruler** from the **View** menu. To change the measurement system used in the ruler, click on the ruler with the right mouse button, and choose **Inches**, **Centimeters**, **Points**, or **Picas** from the pop-up menu. For more information on using the ruler to format your worksheet, refer to "Using the worksheet ruler" on page 79.

Working with Windows

When you start Mathcad, you open up a window on a Mathcad *worksheet*. You can have as many worksheets open as your available system resources allow. This allows you to work on several worksheets at once by simply clicking the mouse in whichever document window you want to work in.

There are times when a Mathcad worksheet cannot be displayed in its entirety because the window is too small. To bring unseen portions of a worksheet into view, you can:

- Make the window larger as you do in other Windows applications.
- Choose **Zoom** from the **View** menu or click 100% on the Standard toolbar nd choose a number smaller than 100%.

You can also use the scroll bars, mouse, and keystrokes to move around the Mathcad window, as you can in your other Windows applications. When you move the mouse pointer and click the mouse button, for example, the cursor jumps from wherever it was to wherever you clicked.

Tip Mathcad supports the Microsoft IntelliMouse and compatible pointing devices. Turning the wheel scrolls the window one line vertically for each click of the wheel. When you press [**Shift**] and turn the wheel, the window scrolls horizontally.

See "Arrow and Movement Keys" on page 314 in the Appendices for keystrokes to move the cursor in the worksheet. If you are working with a longer worksheet, choose **Go to Page** from the **Edit** menu and enter the page number you want to go to in the dialog box. When you click "OK," Mathcad places the top of the page you specify at the top of the window.

Tip Mathcad supports standard Windows keystrokes for operations such as file opening, [**Ctrl**]**O**], saving, [**Ctrl**]**S**], printing, [**Ctrl**]**P**, copying, [**Ctrl**]**C**], and pasting, [**Ctrl**]**V**]. Choose **Preferences** from the **View** menu and check "Standard Windows shortcut keys" in the Keyboard Options section of the General tab to enable all Windows shortcuts. Remove the check to use shortcut keys supported in earlier versions of Mathcad.

Regions

Mathcad lets you enter equations and text anywhere in the worksheet. Each equation, piece of text, or other element is a *region*. Mathcad creates an invisible rectangle to hold each region. A Mathcad worksheet is a collection of such regions. To start a new region in Mathcad:

1. Click anywhere in a blank area of the worksheet. You see a small crosshair.

   Anything you type appears at the crosshair.
2. If the region you want to create is a math region, just start typing anywhere you put the crosshair. By default Mathcad understands what you type as mathematics. See "A Simple Calculation" on page 12 for an example.
3. To create a text region, first choose **Text Region** from the **Insert** menu and then start typing. See "Entering Text" on page 14 for an example.

In addition to equations and text, Mathcad supports a variety of plot regions. See "Graphs" on page 17 for an example of inserting a two-dimensional plot.

Tip Mathcad displays a box around any region you are currently working in. When you click outside the region, the surrounding box disappears. To put a permanent box around a region, click on it with the right mouse button and choose **Properties** from the pop-up menu. Click on the Display tab and click the box next to "Show Border."

Selecting Regions

To select a single region, simply click it. Mathcad shows a rectangle around the region.

To select multiple regions:

1. Press and hold down the left mouse button to anchor one corner of the selection rectangle.
2. Without letting go of the mouse button, move the mouse to enclose everything you want to select inside the selection rectangle.
3. Release the mouse button. Mathcad shows dashed rectangles around regions you have selected.

Tip You can also select multiple regions anywhere in the worksheet by holding down the [**Ctrl**] key while clicking. If you click one region and [**Shift**]-click another, you select both regions and all regions in between.

Moving and Copying Regions

Once the regions are selected, you can move or copy them.

Moving regions

You can move regions by dragging with the mouse or by using **Cut** and **Paste**.

To drag regions with the mouse:

1. Select the regions as described in the previous section.
2. Place the pointer on the border of any selected region. The pointer turns into a small hand.